

Quantum-classical crossover in electrodynamics

Janos Polonyi*

Theoretical Physics Laboratory, Louis Pasteur University, Strasbourg, France

(Dated: August 15, 2006)

A classical field theory is proposed for the electric current and the electromagnetic field interpolating between microscopic and macroscopic domains. It represents a generalization of the density functional for the dynamics of the current and the electromagnetic field in the quantum side of the crossover and reproduces standard classical electrodynamics on the other side. The effective action derived in the closed time path formalism and the equations of motion follow from the variational principle. The polarization of the Dirac-see can be taken into account in the quadratic approximation of the action by the introduction of the displacement field strengths as in conventional classical electrodynamics. Decoherence appears naturally as a simple one-loop effect in this formalism. It is argued that the radiation time arrow is generated from the quantum boundary conditions in time by decoherence at the quantum-classical crossover and the Abraham-Lorentz force arises from the accelerating charge or from other charges in the macroscopic or the microscopic side, respectively. The functional form of quantum renormalization group, the generalization of the renormalization group method for the density matrix, is proposed to follow the scale dependence through the quantum-classical crossover in a systematical manner.

PACS numbers: 03.50.De, 05.10.Cc, 12.20.-m

I. INTRODUCTION

Classical systems traditionally serve as starting points for quantization. But the opposite order, namely the construction of macroscopic physics from quantum principle, is needed to understand the great division line in Physics, the quantum-classical crossover. One expects no surprise from the traditional way of thinking because the usual equations of motions are supposed to be recovered in the narrow wave-packet limits according to Ehrenfest theorem. Though formally correct, this theorem does not guarantee that the original, naive scenario of classical particles or bodies is recovered because the relativistic and many-body aspects of quantum physics introduce correlations in the narrow wave-packet limit which render the picture of classically localized particles and bodies more involved. The goal of this paper is to derive classical dynamics, namely action and variational equations of motion for excitations in QED.

The dynamics of expectation values can be obtained by means of the closed time path (CTP) formalism proposed by Schwinger long time ago [1]. One expects that the dynamics for the electric current and the electromagnetic field will be governed by the action

$$S = - \sum_i M_i \int_{x_i} ds - \sum_i e_i \int_{x_i} A_\mu(x) dx^\mu - \frac{1}{4} \int d^4x (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2 + S_Q \quad (1)$$

involving the world lines $x_i(s)$ of charge e_i , mass M_i and the vector potential $A_\mu(x)$ and S_Q standing for the corrections due to quantum fluctuations. But there is a negative result conjecturing that no variational principle is available for the derivation of the equations of motion [2] and in fact, up to our knowledge no positive result had been communicated in this direction ever since. It will be shown below that the action (1) together with its variational principle can actually be derived. The term S_Q in Eq. (1) represents the systematical improvement of classical electrodynamics by taking into account quantum effects arising from the polarization of the Dirac-see without interaction, vacuum polarizations due to electromagnetic interactions and boundary conditions in time.

Once the action governing the expectation values is constructed a number of interesting questions open up. The first set of questions concerns the form of the action. How can we separate off the many-body aspects from a particle propagating in the non-interacting Dirac-see and what corrections will be added to the free mechanical action, the first term on the right hand side of Eq. (1)? The dynamics of the expectation values of local operators should give a better insight into the quantum-classical transition regime because the space-time resolution of the expectation value of a local operator is limited by the UV cutoff only. How can we separate microscopic and macroscopic effects in the

*Electronic address: polonyi@fresnel.u-strasbg.fr

expectation values, or preferably in the action? The effects of the polarization of the Dirac-see are included in the quantum corrections to the action. Does this happen in the usual manner polarization is taken into account in classical electrodynamics? Another set of questions arise about the radiational time arrow. The boundary conditions in time are realized on the microscopic level in QED and appear as infinitesimal imaginary contributions in S_Q . How does the experimentally well established retarded Green function arise from the time reversal invariant dynamics? Finally, such a calculation addresses the measurement theory of Quantum Mechanics. In fact, in analyzing the measurement process one always ends up with the study of the interactions between a small and a large system. Are there some peculiarities in the dynamics when the large parameter, the ratio of the number of degrees of freedom of the large and the small system, diverges, like spontaneous symmetry breaking occurring in some models in the thermodynamical limit? We attempt to give below at least some indications about the answers.

The CTP formalism has originally been introduced in Quantum Field Theory [1–12] for the description of the time dependence of expectation values. Similar scheme was developed independently in the framework of Quantum Mechanics, too. The formal invariance of transition probabilities under time inversion lead to the construction of time symmetrical Quantum Mechanics [13, 14]. The explicit appearance of the density matrix in the formalism explains the applications of this scheme in calculating the reduced density matrix in Quantum Mechanics [15] with special attention paid to dissipative processes [16–20]. Furthermore, the description of decoherence [21–24] in the framework of consistent histories [25–27] can be achieved in a natural manner [28–30] in the CTP formalism. Finally, one should mention promising applications of this scheme to cosmological problems [31–34], to kinetic Quantum Field Theory [35–38] and to the renormalization group [39, 40].

In these applications of the CTP formalism the degrees of freedom are doubled, due to the systematic implementation of time evolution in the Heisenberg picture and/or to the canonical description of dissipative processes. But this doubling remains rather formal from the point of view of classical physics. The present work is based on the observation that one can actually construct variational principle and canonical dynamics for the expectation values of observables within the CTP formalism. The choice of the microscopic degrees of freedom which become classical in the macroscopic region is unique up to a parameter which is allowed due to the unitarity of the time evolution for closed systems. The other, independent combination of the doubled degrees of freedom takes care of the coupling of the system to its environment and becomes suppressed in the macroscopic region. Such an explicit splitting of the variables offers a new, more natural starting point for the exploration of the quantum-classical crossover, presented in this paper within the framework of QED. The scope of this work is restricted to general, qualitative issues, such as the current dynamics in the Dirac-see, the polarizability of the vacuum, the radiation time arrow, the decoherence of non-relativistic charges and the renormalization group scheme for the density matrix to describe the quantum-classical crossover. The classical dynamics derived in these contexts yields new results already at such a qualitative level. We plan to present the detailed, quantitative studies of these and related issues in subsequent publications.

The organisation of this paper is the following. The action for the expectation values and some partial answer to these questions will be found within the CTP formalism, introduced in Section II with special attention paid to the time evolution of the averages of observables and the role of the boundary conditions in time. The simplest system where the role of the boundary conditions in time can be traced in forming the time arrow is that of non-interacting particles. Section III contains the derivation of the action for the expectation value of the free photon field. The electric current is a composite operator and its expectation value is controlled by a substantially more complicated action. Section IV is devoted to these complications and the presentation of the quadratic approximation to the action for the current. The case of interacting electrons and photons is considered in Section V. The use of classical expectation values, obtained with resolutions belonging to the quantum domain, in particular decoherence, radiational time arrow and vacuum polarization are commented on briefly in Section VI. The systematic study of the dependence of the expectation values on the space-time resolution around the quantum-classical crossover can be achieved by means of the extension of the renormalization group strategy for the density matrix. Some qualitative remarks about the renormalized trajectory of this scheme are presented in Section VII. Finally, Section VIII is reserved for the summary of the results. The technical parts of the calculation are collected in appendices. The two-point function of a hermitian local operator of the CTP formalism is introduced in Appendix A and some details of the calculation of the two-point vertex function in the non-interacting Dirac-see are collected in Appendix B. In Appendix C the expressions needed for the connected two-point functions for the current and the photon field are presented. The calculation of the effective action for the expectation values of these operators is outlined in Appendix D. The evolution equation for the functional renormalization group in the CTP formalism is derived in Appendix E.

II. EXPECTATION VALUES IN QUANTUM FIELD THEORY

Our goal is to establish relations among the expectation values of a set of local observables like $\langle \Psi(t) | O_a(\mathbf{x}) | \Psi(t) \rangle_S$ given in the Schrödinger representation. The problem is interesting when the system is in an excited state. The only

technical restriction on the nature of the excitations is that the initial state should be the result of an adiabatic time evolution generated from the ground state by adding the source term $\sum_a j_a(t, \mathbf{x}) O_a(\mathbf{x})$ to the Hamiltonian density where the summation is over the observables considered. Closed systems will be considered below and the issue of relaxation and equilibrium in the presence of a reservoir will not be addressed. The state vector of the system $|\Psi(t)\rangle_S = U(t, t_i)|\Psi_i\rangle_S$ is given in terms of the initial condition $|\Psi_i\rangle_S$ imposed at time t_i and the time evolution operator $U(t, t_i) = \exp -i(t - t_i)H$ involving the Hamiltonian H . Such expectation values are usually given in the Heisenberg representation, $\langle\Psi_i|A(x)|\Psi_i\rangle_H$, where $x = (t, \mathbf{x})$ and $A(x) = U(t_i, t)A(\mathbf{x})U(t, t_i)$ and their perturbation series is obtained in terms of Green functions which are different from the usual ones occurring in the expressions of scattering amplitudes. Although the trajectory $|\Psi(t)\rangle_S$ is fixed in the space of states by the initial condition $|\Psi_i\rangle_S$ the superposition principle allows us to project the system at time t_f on a given final state $|\Psi_f\rangle_S$ and to define the scalar product $\langle\Psi_f|\Psi(t)\rangle_S = \langle\Psi_f|U(t_f, t_i)|\Psi(t)\rangle_S$ interpreted as the transition amplitude between initial and final states. Its perturbative series contain matrix elements like $\langle\Psi_f|U(t_f, t)A(\mathbf{x})U(t, t_i)|\Psi_i\rangle_S$ which are not interpretable in terms of measurements according to standard rules of Quantum Mechanics. Such matrix elements are reduced to an expectation value for the trivial case $|\Psi_f\rangle = |\Psi_i\rangle = |0\rangle$ only (by setting the ground state energy to zero).

Let us return to our problem, the calculation of the expectation values in excited states and consider the generalization of the forward scattering amplitude for several observables,

$$\langle\Psi_i|T[U(t_f, t_i)A_n(x_n) \cdots A_1(x_1)]|\Psi_i\rangle_H = \langle\Psi_i|T[U(t_f, x_n^0)A_n(\mathbf{x}_n)U(x_n^0, t_i) \cdots U(t_i, x_1^0)A_1(\mathbf{x}_1)U(x_1^0, t_i)]|\Psi_i\rangle_S \quad (2)$$

which differs from the Green function defined as the expectation value of the operator $T[A_n(x_n) \cdots A_1(x_1)]_H$, [1],

$$\langle\Psi_i|T[A_n(x_n) \cdots A_1(x_1)]|\Psi_i\rangle_H = \langle\Psi_i|U(t_i, t_f)T[U(t_f, x_n^0)A_n(\mathbf{x}_n)U(x_n^0, t_i) \cdots U(t_i, x_1^0)A_1(\mathbf{x}_1)U(x_1^0, t_i)]|\Psi_i\rangle_S. \quad (3)$$

The difference occurs in the way the scattering matrix $U(t_f, t_i)$ appears in these expressions. In Eq. (2) the time evolution is constrained in such a manner that the system arrives at the state $|\Psi_i\rangle$ at time t_f , in contrast to Eq. (3) where the time evolution is open ended, without any constraint on the evolution after the action of the observables. The functions given by Eqs. (2) and (3) are identical for $|\Psi_i\rangle = |0\rangle$ because the vacuum is stable during the time evolution. But the dynamics of expectation values in an excited state, $|\Psi_i\rangle \neq |0\rangle$, requires the use of the expectation value (3) obtained in the closed time path (CTP) formalism [2–12], rather than the contribution (2) to the scattering amplitude. The careful reader may object that it is sufficient to use the vacuum state because all physical states can be obtained from the vacuum by applying local excitations, ie. $|\Psi_i\rangle_H = A_i(x_i)|0\rangle_H$ where $A_i(x)$ is a local operator in space-time. Extend the time evolution in this case in such a manner that the system starts at time $t = t_0 < t_i$ and ends at time $t = t_1 > t_f$ with the vacuum state and write Eq. (3) as

$$\langle 0|\bar{T}[A_i(x_i)U(t_0, t_1)]T[U(t_1, x_n^0)A_n(\mathbf{x}_n)U(x_n^0, t_i) \cdots U(t_i, x_1^0)A_1(\mathbf{x}_1)U(x_1^0, t_i)A_i(x_i)U(t_i, t_0)]|0\rangle_S, \quad (4)$$

where \bar{T} denotes anti-time ordering. The comparison of this expression with (2) shows that the basic difference between the CTP formalism and the usual scattering amplitudes is the presence of the anti-time ordered piece in the former case. The proper treatment of such matrix elements requires the introduction of independent time variables for the time ordered and the anti-time ordered operators.

To compare the expectation values obtained by constrained and open ended time evolutions we slightly generalize the CTP formalism and allow that the system be described by the density matrices ρ_i and ρ_f in the initial and the final states, respectively. Furthermore, we introduce two sets of external sources, $j_a^\pm(x)$, coupled linearly to a number of local observables, $O_a(\mathbf{x})$, in the hermitian Hamiltonian which is extended to

$$H^\pm(t) = H \mp \sum_a \int d^3x j_a^\pm(t, \mathbf{x}) O_a(\mathbf{x}). \quad (5)$$

Finally, we define the generating functional for the connected Green functions

$$e^{iW[j^+, j^-]} = \text{Tr} \bar{T}[e^{i \int_{t_i}^{t_f} dt H^-(t)}] \rho_f T[e^{-i \int_{t_i}^{t_f} dt' H^+(t')}] \rho_i \quad (6)$$

given in the Schrödinger representation. The sources play a double role made explicit by the parameterization $j^\pm = j/2 \pm \bar{j}$. On the one hand, the variation of j is used to generate the desired expectation values, and on the other, \bar{j} drives the system adiabatically from the vacuum at time $t = t_0$ to the desired initial state $|\Psi_i\rangle$ at $t = t_i$. This allows us to set $\rho_i = |0\rangle\langle 0|$ at $t_0 = -\infty$ with the price of having non-vanishing physical external sources, \bar{j} , for some initial times $t < t_i$. The quantity (6) is the transition probability between the states specified by the density matrices ρ_i and ρ_f for physical external sources. Expressions like (6) have already been used a number of times with O_a chosen to be projection operator, eg. for the time-symmetric formulation of Quantum Mechanics [13, 14] and for the study of histories [25–27].

It is natural to introduce the parameterization

$$\phi^\pm = \phi \pm \frac{\phi^{\text{adv}}}{2} \quad (7)$$

for the field variables. The measured expectation values of the field at a given time t involve the diagonal part of the functional $\langle \phi^+ | \rho_i(t) | \phi^- \rangle$ with $\phi^{\text{adv}} = 0$. The canonical momenta, represented by the operators $\Pi^\pm = -i\delta/\delta\phi^\pm$, use a slightly extended domain, $\phi^{\text{adv}} \approx 0$, of the density matrix.

There are two time axis in the CTP formalism and each degree of freedom exists in two copies, realized first in the case of thermal equilibrium, $\rho_i = Z^{-1} \exp -H/T$, [41–45], but can also be clearly seen in the path integral representation

$$e^{iW[j^+, j^-]} = \int D[\phi^+] D[\phi^-] e^{iS[\phi^+] - iS^*[\phi^-] + \sum_\sigma \int_{t_i}^{t_f} dt \sum_a \int d^3x j_a^\sigma(t, \mathbf{x}) O_a(\phi^\sigma(t, \mathbf{x}))}] \rho_f[\phi^-(t_f), \phi^+(t_f)] \rho_i[\phi^+(t_i), \phi^-(t_i)] \quad (8)$$

given as a functional integration over the trajectories ϕ^σ , $\sigma = \pm$, with the density matrix elements $\langle \phi^+ | \rho | \phi^- \rangle = \rho[\phi^+, \phi^-]$ in the integrand. The duplication is made explicit by defining two kinds of averages, one for each time axis,

$$\langle \bar{T}[A], T[B] \rangle_{\rho_f, \rho_i}^{j, \bar{j}} = \text{Tr} \bar{T}[A e^{i \int_{t_i}^{t_f} dt H^-(t)}] \rho_f T[B e^{-i \int_{t_i}^{t_f} dt H^+(t')}] \rho_i. \quad (9)$$

Notice that $\langle \mathbb{1}, T[B] \rangle_{\rho_f, \rho_i}^{0, \bar{j}}$ is real according to the relation

$$(\langle \bar{T}[A], T[B] \rangle_{\rho_f, \rho_i}^{j, \bar{j}})^* = \langle \bar{T}[B], T[A] \rangle_{\rho_f, \rho_i}^{-j, \bar{j}}, \quad (10)$$

showing an inherent time inversion invariance ie. the time inversion leaves ρ_i , ρ_f and \bar{j} invariant and flips the sign of the book-keeping variable j only. The time-dependence of the density matrices

$$\begin{aligned} \rho_i(t) &= T[e^{-i \int_{t_i}^t dt' H^+(t')}] \rho_i \bar{T}[e^{i \int_{t_i}^t dt H(t)}], \\ \rho_f(t) &= \bar{T}[e^{i \int_t^{t_f} dt H(t)}] \rho_f T[e^{-i \int_t^{t_f} dt' H^+(t')}] \end{aligned} \quad (11)$$

allows us to write

$$e^{iW[j^+, j^-]} = \text{Tr} \rho_f(t) \rho_i(t), \quad (t_i \leq t \leq t_f). \quad (12)$$

The right hand side of this equation is the scalar product of the initial and final hermitean density matrices taken at an arbitrary time indicating that the CTP formalism is based on the transition probabilities rather than amplitudes and is time reversal invariant. The duplication of the time variables represents the independence of the quantum fluctuations in the bras and the kets of the density matrices. We shall argue below that these fluctuations become correlated and the usual single time axis formalism is recovered in the macroscopic limit when decoherence [21, 24] suppresses the off-diagonal elements of the density matrix.

It will be instructive to follow the dynamics in the presence of two different boundary conditions in time. The fixed boundary condition (FBC) for pure initial and final states, defined by $\rho_i = |\Psi_i\rangle\langle\Psi_i|$, $\rho_f = |\Psi_f\rangle\langle\Psi_f|$ decouples the dynamics of the two time axis. The additivity of transition probabilities yields

$$\langle \mathbb{1}, \mathbb{1} \rangle_{\rho_f^{(1)} + \rho_f^{(2)}, \rho_i}^{0, \bar{j}} = \langle \mathbb{1}, \mathbb{1} \rangle_{\rho_f^{(1)}, \rho_i}^{0, \bar{j}} + \langle \mathbb{1}, \mathbb{1} \rangle_{\rho_f^{(2)}, \rho_i}^{0, \bar{j}} \quad (13)$$

for $\text{Tr} \rho_f^{(1)} \rho_f^{(2)} = 0$. The open boundary condition (OBC) defined by $\rho_i = |\Psi_i\rangle\langle\Psi_i|$ and $\rho_f = \sum_n |\Psi^{(n)}\rangle\langle\Psi^{(n)}| = \mathbb{1}$, $\{|\Psi^{(n)}\rangle\}$ being a basis for the states corresponds to unconstrained time evolution and couples the dynamics of the two time axis at $t = t_f$. Note that the time t_f at which the final condition $\rho_f = \mathbb{1}$ is imposed can be chosen arbitrarily for unitary time evolution as long as it is later than the time for which the latest observable inserted. We call a boundary condition reflecting if the time at which it is imposed as an initial or final condition influences the measurable expectation values. The open boundary condition is non-reflecting because $\rho_f = \mathbb{1}$ commutes with all observables.

The main virtue of the Heisenberg representation and the CTP formalism in particular, is to render the initial condition problems of Quantum Mechanics simple. The hyperbolic Schrödinger equation allows us to solve the initial condition problem in terms of state vectors but these are in general complicated objects. We can deal more efficiently with numbers, for instance matrix elements and therefore it is advantageous to convert operator equations into c-number equations. But a matrix element contains a bra and a ket, corresponding to the final and initial states,

respectively as in Eq. (2) and it is not clear how to express the solution of the initial condition problem of the Schrödinger equation with open ended time evolution in terms of such matrix elements. This is the problem which is solved in the framework of the CTP formalism by extending ρ_i and ρ_f to positive semidefinite hermitean operators beyond the domain of density matrices.

The direction of the time is determined by the phase of an eigenstate of the Hamiltonian as a function of the time, and the time runs in opposite directions along the two time axis. Due to the presence of both time directions there is no explicit time arrow in the expectation values which contain both retarded and advanced effects. But one can separate these effects, as far as the external sources are concerned, in the case of open boundary condition and we shall see below that ϕ and ϕ^{adv} of Eq. (7) are the dynamical variables which are build up by retarded and advanced effects of the external sources, respectively.

An important property of the open boundary condition with unitary time evolution is that the generating functional is vanishing for physical external sources, $W[j, -j] = 0$. As a result we have to vary non-physical sources, \bar{j} , to generate the measured expectation values $\langle O_a(x) \rangle = \langle \mathbb{1}, O_a(x) \rangle = \langle O_a(x), \mathbb{1} \rangle$. This suggests the slight generalization of the parameterization of the sources, $\bar{j} \rightarrow \bar{j} + \kappa j$, ie.

$$j^\pm = \frac{j}{2}(1 \pm \kappa) \pm \bar{j}, \quad (14)$$

giving

$$\langle O_a(x) \rangle = \frac{\delta W[j, \bar{j}]}{\delta j_a(x)} \Big|_{j=0} = \frac{1+\kappa}{2} \langle \mathbb{1}, O_a(x) \rangle + \frac{1-\kappa}{2} \langle O_a(x), \mathbb{1} \rangle \quad (15)$$

for arbitrary choice of κ . The κ -independence of the expectation values reflects a one dimensional symmetry of the physical sector of the CTP formalism.

The parameter κ appearing here mixes the non-diagonal quantum fluctuations into the expectation value of the observables. We need $\kappa \neq 0$ ie. both kind of fluctuations, to derive variational equations of motion for the measured expectation values. This condition seems natural because both canonical variables are needed in the dynamics but the source j is coupled only to the diagonal quantum fluctuations for $\kappa = 0$ which are not sufficient to reconstruct the expectation values of the canonical momenta Π . The detailed argument goes as follows. The action constructed for the field variable ϕ is obtained in terms of connected Green functions. The n -point functions give rise to $O(\phi^{n-1})$ terms in the equation of motion. The simplest linearized equation of motion arises from some connected two-point functions. These two-point functions must include field variables at different times to generate time evolution for the expectation values. Let us consider the combination

$$G = \frac{1}{2\Delta t} \langle \Psi_i | \phi(t, \mathbf{x}) [\phi(t, \mathbf{y}) - \phi(t - \Delta t, \mathbf{y})] | \Psi_i \rangle_H \quad (16)$$

of two-point functions with small Δt as a typical term which can also be written as

$$G = -\frac{i}{2} \langle \Psi_i | \phi(t, \mathbf{x}) [\phi(t, \mathbf{y}), H] | \Psi_i \rangle_H + O(\Delta t) \quad (17)$$

according to Eq. (3). The kinetic energy $\Pi^2(t, \mathbf{x})/2$ in the Hamiltonian density yields

$$G = \langle \Psi_i | \phi(t, \mathbf{x}) \Pi(t, \mathbf{y}) | \Psi_i \rangle_H + O(\Delta t), \quad (18)$$

and shows that the appearance of the canonical momentum operator is unavoidable in the Green functions involving fields at different times. We have the option of keeping in the formalism the expectation values of both canonical variables, ϕ and Π , the resulting variational equations being the quantum analogies of the Hamilton equations. But once we have decided to retain the coordinate averages only we need $\kappa \neq 0$ to couple both diagonal and off-diagonal fluctuations to the source j . The off-diagonal fluctuations drop out from the field average (15) but the $O(j^2)$ terms in the generating functional $W[j, \bar{j}]$ and the $O(\phi^2)$ term of the action will retain them.

A simpler but more formal reasoning starts with the identity $W^*[j^+, j^-] = -W[-j^-, -j^+]$ obtained by comparing $W[j^+, j^-]$ with its complex conjugate in Eq. (6). This identity reads as $W^*[j, \bar{j}] = -W[-\bar{j}, j]$ for $\kappa = 0$, indicating that the real part of $W[j, \bar{j}]$ which will be important for the equation of motion is an odd functional of j and has, in particular an $O(j^2)$ part. As a result the equation of motion for ϕ contains even powers of the field variables. In order to have linear part in the equation of motion we have to allow $\kappa \neq 0$.

As a simple demonstration that the measured expectation value results from the retarded effects of the classical external sources we consider the linear response formulae. A physical external source, $j_+^a(x) = -j_-^a(x) = \bar{j}^a(x)$ drives

the time evolution of the expectation value of the operator $O_\ell(x)$ in the perturbation expansion of the external source according to

$$\begin{aligned} \langle \mathbb{1}, O_\ell(x) \rangle_{\mathbb{1}, \rho_i}^{0, \bar{j}} &= -i \frac{\delta}{\delta j_\ell^+(x)} \sum_{n_+, n_- = 0}^{\infty} \frac{(-1)^{n_-}}{n_+! n_-!} \left(\sum_a \int dx' j^a(x') \frac{\delta}{\delta j_a^+(x')} \right)^{n_+} \\ &\quad \times \left(\sum_b \int dx'' j^b(x'') \frac{\delta}{\delta j_b^-(x'')} \right)^{n_-} e^{iW[j^+, j^-]}|_{j^+ = -j^- = \bar{j}}. \end{aligned} \quad (19)$$

The linear response formula for $O_\ell(t)$ is the $O(\bar{j})$ part of the right hand side,

$$i \int dx' \sum_a \bar{j}_a(x') \left(\langle O_a(x'), O_\ell(x) \rangle_{\mathbb{1}, \rho_i}^{0,0} - \langle \mathbb{1}, T[O_a(x') O_\ell(x)] \rangle_{\mathbb{1}, \rho_i}^{0,0} \right), \quad (20)$$

which can be written as

$$\begin{aligned} &i \int d^3x' \int_{t_i}^{t_f} dt' \sum_a \bar{j}_a(t', \mathbf{x}') \Theta(t' - t) \left(\langle O_a(x'), O_\ell(x) \rangle_{\mathbb{1}, \rho_i}^{0,0} - \langle \mathbb{1}, T[O_a(x') O_\ell(x)] \rangle_{\mathbb{1}, \rho_i}^{0,0} \right) \\ &+ i \int d^3x' \int_{t_i}^{t_f} dt' \sum_a \bar{j}_a(t', \mathbf{x}') \Theta(t - t') \left(\langle O_a(x'), O_\ell(x) \rangle_{\mathbb{1}, \rho_i}^{0,0} - \langle \mathbb{1}, T[O_a(x') O_\ell(x)] \rangle_{\mathbb{1}, \rho_i}^{0,0} \right). \end{aligned} \quad (21)$$

If the final time boundary condition is non-reflecting and can be imposed at any time superior to t without modifying the expectation values then the relation

$$\Theta(t' - t) \langle O_a(x'), O_\ell(x) \rangle_{\mathbb{1}, \rho_i}^{0,0} = \Theta(t' - t) \langle \mathbb{1}, T[O_a(x') O_\ell(x)] \rangle_{\mathbb{1}, \rho_i}^{0,0} \quad (22)$$

holds. In fact, one can reduce t_f down to t' followed by the commutation of the perturbation $O_a(x')$ with ρ_f and the placing of it at the end of the anti-time ordered part of the expression. This identity shows the cancellation of the advanced part of the causal propagator and explains that the usual retarded response formulae are the result of the interference between the dynamics evolving along the two different time paths. Whenever a final boundary condition is used which can not be displaced in time because it does not commute with all operators in question then advanced effects are left behind. Notice that this remains valid in any order. In fact, the possibility of reducing t_f until it reaches the time of observation, t , removes any influence of the sources on the expectation value after the time of the measurement. The condition of the cancellation was that the non-vanishing sources are classical, $j^+ = -j^-$. The non-classical part of the source, j , representing a coupling among the degrees of freedom of a closed quantum system may induce advanced effects. But this component is suppressed in the macroscopic limit by decoherence.

III. PHOTONS

The simplest context in which the dynamics of the field expectation values can be studied is the case of free photons. The expectation values are obtained in two steps. First the generating functional for the connected Green functions of the conserved current is calculated. This produces the expectation values in terms of the external sources which drive the system to the desired initial condition. In the second step a number of different effective actions and their variational equations of motion are constructed by performing a Legendre transformation on the external sources.

A. Photon propagator

The generating functional for the connected Green functions is constructed by coupling an external current to the photon field and using the pair of extended Hamiltonians $H \rightarrow H^\pm(t) = H \mp \int d^3x j^{\pm\mu}(t, \mathbf{x}) A_\mu(\mathbf{x})$ in Eq. (6). It is advantageous to carry out the calculations in the path integral formalism where the action for the photon field is written as

$$\begin{aligned} S_M[A] &= \int_x \left[-\frac{1}{4} (\partial_\mu A_{\nu,x} - \partial_\nu A_{\mu,x})^2 - \frac{\xi}{2} (\partial^\mu A_{\mu,x})^2 \right] \\ &= \frac{1}{2} A \cdot D_0^{-1} \cdot A \end{aligned} \quad (23)$$

Here D_0 is the free photon propagator $D_0^{-1} = D_T^{-1} + D_L^{-1}$ and the projection operators onto the transverse and longitudinal components of the photon field $T^{\mu\nu} = g^{\mu\nu} - L^{\mu\nu}$ and $L^{\mu\nu} = \partial^\mu \partial^\nu / \square$ are used to construct $D_T^{-1} = (\square - i\epsilon)T$ and $D_L^{-1} = \xi(\square - i\epsilon)L$.

A few words about the notations: The field configurations $\phi(x)$ are usually considered as vectors, ϕ_x , and space-time integrals with occasional summation over repeated indices as scalar products, eg. $\int dx \phi(x) \chi(x) = \int_x \phi_x \chi_x = \phi \cdot \chi$ or $\int dx A_\mu(x) j^\mu(x) = A \cdot j$. The space-time coordinates are written as $x = (t, \mathbf{x})$, the space-time indices are given by Greek letters, the Latin letters denote combined indices like $a = (\mu, x)$. Repeated indices are summed/integrated over.

The generating functional $W^{\text{phot}}[j^+, j^-]$ is given in the framework of the path integral representation as

$$e^{iW^{\text{phot}}[\hat{j}]} = \int D[\hat{A}] e^{\frac{i}{2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} + i \hat{j} \cdot \hat{A}}, \quad (24)$$

where the two component integral variables $\hat{A} = (A^+, A^-)$ and external sources $\hat{j} = (j^+, j^-)$ are introduced to simplify the expressions. The current j^\pm will be parameterized as in Eq. (14). Some complications arise due to the presence of the density matrices ρ_i and ρ_f in the functional integral. Anticipating the perturbation expansion we use Gaussian density matrices which generate quadratic expressions in the action of the path integral, ie. $\ln \rho_i[A_{+,t_i}; A_{-,t_i}]$ and $\ln \rho_f[A_{-,t_f}; A_{+,t_f}]$ should be at most quadratic in the fields. The inverse block propagator is

$$\hat{D}_0^{-1} = \begin{pmatrix} D_0^{-1} & 0 \\ 0 & -D_0^{-1*} \end{pmatrix} + \hat{D}_{\text{BC}}^{-1}, \quad (25)$$

where the second term on the right hand sides stands for the contributions of the density matrices and is non-vanishing for $t = t_i$ or t_f only. Notice the presence of an infinitesimal imaginary part in the inverse propagator (25). The operator expression (6) contains a hermitian Hamiltonian because the (anti)time ordering implies the time boundary conditions instruction. In contrast, the time boundary conditions involve the infinitesimal imaginary parts in the inverse propagators in the path integral representation.

Most of the work reported below is based on OBC with the vacuum as initial state. Although one can determine \hat{D}_0^{-1} directly in the path integral formalism for this boundary condition it is simpler to construct the photon propagator, \hat{D}_0 , within the operator formalism. The result is

$$\hat{D}_0 = \begin{pmatrix} D_0^{\text{near}} + i\Im D_0 & -\frac{1}{2} D_0^{\text{far}} + i\Im D_0 \\ \frac{1}{2} D_0^{\text{far}} + i\Im D_0 & -D_0^{\text{near}} + i\Im D_0 \end{pmatrix} \quad (26)$$

in the base (j^+, j^-) where D^{near} and D^{far} denote the near and far field Green functions [46], respectively and $i\Im D_0$ stands for the imaginary part of the causal (Feynman) Green function. The retarded and advanced Green functions, $D_0^{\text{ret}} = D_0^{\text{near}} + \frac{1}{2} D_0^{\text{far}}$ and $D_0^{\text{adv}} = D_0^{\text{near}} - \frac{1}{2} D_0^{\text{far}}$ will frequently be used, as well, cf. Appendix A.

The Gaussian integral can easily be carried out in Eq. (24) yielding

$$W^{\text{phot}}[\hat{j}] = -\frac{1}{2} \hat{j} \cdot \hat{D}_0 \cdot \hat{j} \quad (27)$$

and a straightforward calculation gives

$$W_{\text{OBC}}^{\text{phot}}[j, \bar{j}] = -\frac{1}{2} \begin{pmatrix} \bar{j} & j \end{pmatrix} \cdot \begin{pmatrix} 0 & D_0^{\text{adv}} \\ D_0^{\text{ret}} & \kappa D_0^{\text{near}} + i\Im D_0 \end{pmatrix} \cdot \begin{pmatrix} \bar{j} \\ j \end{pmatrix} \quad (28)$$

for open boundary condition. In the case of fixed boundary condition the two time axis decouple yielding

$$\begin{aligned} W_{\text{FBC}}^{\text{phot}}[\hat{j}] &= -\frac{1}{2} j^+ \cdot (D_0^{\text{near}} + i\Im D_0) \cdot j^+ + \frac{1}{2} j^- \cdot (D_0^{\text{near}} - i\Im D_0) \cdot j^- \\ &= -\frac{1}{2} \begin{pmatrix} \bar{j} & j \end{pmatrix} \cdot \begin{pmatrix} 2i\Im D_0 & D_0^{\text{near}} + \kappa i\Im D_0 \\ D_0^{\text{near}} + \kappa i\Im D_0 & \kappa D_0^{\text{near}} + \frac{1+\kappa^2}{2} i\Im D_0 \end{pmatrix} \cdot \begin{pmatrix} \bar{j} \\ j \end{pmatrix}. \end{aligned} \quad (29)$$

B. Effective action

The role of the effective action is to provide the functional for the variational equations satisfied by the space-time dependent expectation values and it is introduced by performing a functional Legendre transformation on the

generating functional of connected Green functions. Special attention must be paid to the fact that the Legendre pair of the sources, the expectation values, are complex in general. Accordingly we perform the Legendre transformation separately for the real and imaginary part of $W[\hat{j}] = \Re W[\hat{j}] + i\Im W[\hat{j}]$. The measured expectation value of the photon field is always real and it will be determined by $\Re W[\hat{j}]$. Therefore, we start with the effective action

$$\Gamma^{\text{phot}}[A, A^{\text{adv}}] = \Re W^{\text{phot}}[j, \bar{j}] - \bar{j} \cdot A^{\text{adv}} - j \cdot A, \quad (30)$$

with independent variables consisting of the measured expectation value of the photon field

$$A = \frac{\delta \Re W^{\text{phot}}[j, \bar{j}]}{\delta j}, \quad (31)$$

and an auxiliary field variable

$$A^{\text{adv}} = \frac{\delta \Re W^{\text{phot}}[j, \bar{j}]}{\delta \bar{j}}. \quad (32)$$

The inverse Legendre transform based on the relations (30)

$$\begin{aligned} j &= -\frac{\delta \Gamma^{\text{phot}}[A, A^{\text{adv}}]}{\delta A}, \\ \bar{j} &= -\frac{\delta \Gamma^{\text{phot}}[A, A^{\text{adv}}]}{\delta A^{\text{adv}}} \end{aligned} \quad (33)$$

serves as equations of motion for the expectation values. Since the averages A and A^{adv} receive contributions from the diagonal and non-diagonal fluctuations, respectively, these equations of motion control the time dependence for both the coordinates and the momenta.

The generating functional (28) yields

$$\Gamma_{\text{OBC}}^{\text{phot}}[A, A^{\text{adv}}] = -A^{\text{adv}} \cdot D_0^{\text{ret}-1} \cdot A + \frac{\kappa}{2} A^{\text{adv}} \cdot D_0^{\text{ret}-1} \cdot D_0^{\text{near}} \cdot D_0^{\text{adv}-1} \cdot A^{\text{adv}}, \quad (34)$$

for open boundary condition. The corresponding equations of motion for A and A^{adv} are

$$A^{\text{adv}} = D_0^{\text{adv}} j, \quad (35)$$

and

$$A = D_0^{\text{ret}} \bar{j} + \kappa D_0^{\text{near}} j, \quad (36)$$

respectively. They show that non-diagonal fluctuations contribute to the advanced field A^{adv} generated by j and the physical expectation value, A , is indeed the retarded field created by the physical external current \bar{j} . The two fields, A and A^{adv} remain decoupled in the absence of interactions. For fixed boundary conditions, Eq. (29), we have

$$\Gamma_{\text{FBC}}^{\text{phot}}[A, A^{\text{adv}}] = -A^{\text{adv}} \cdot D_0^{\text{near}-1} \cdot A + \frac{\kappa}{2} A^{\text{adv}} \cdot D_0^{\text{near}-1} \cdot A^{\text{adv}}. \quad (37)$$

This expression can be obtained from Eq. (34) by the replacements $D_0^{\text{ret}} \rightarrow D_0^{\text{near}}$ and $D_0^{\text{adv}} \rightarrow D_0^{\text{near}}$ which represent the loss of the interference between the two time axis.

The effective actions introduced so far provide the equations of motion for the expectation values for both field variables of the CTP formalism. But the field A^{adv} is not physical and its presence is not necessary to extract the time dependence of the photon field A . We simplify the Legendre transformation by keeping \bar{j} as a fixed parameter. The resulting effective action for the photon field alone is

$$\Gamma^{\text{phot}}[A] = \Re W^{\text{phot}}[j, \bar{j}] - j \cdot A, \quad (38)$$

where the dependence of $\Gamma^{\text{phot}}[A]$ on \bar{j} is not shown explicitly. The effective action for OBC turns out to be

$$\Gamma_{\text{OBC}}^{\text{phot}}[A] = -\frac{1}{2\kappa} (A - \bar{j} D_0^{\text{adv}}) \cdot D_0^{\text{near}-1} \cdot (A - D_0^{\text{ret}} \bar{j}), \quad (39)$$

generating an equation of motion $\delta \Gamma^{\text{phot}}[A] / \delta A = -j$ identical to Eq. (36).

So far the value of the parameter κ entering by expression (14) is arbitrary. The choice $\kappa = 0$ seems to be natural and simple and the corresponding currents \bar{j} and j generate retarded or advanced Liénard-Wiechert potentials, respectively, according to the equations of motion. But the effective action (34) contains no $O(A^2)$ piece when $\kappa = 0$ in agreement with the general remark made in Section II stating that the non-physical field can not be eliminated and no classical action and variation principle can be found for the measured expectation value of the photon field. This is reflected in the appearance of the coefficient $1/\kappa$ in the effective action (38) because for $\kappa = 0$ the generating functional (28) has no $O(j^2)$ term in the real part and the Legendre transformation is not defined for linear functions.

For OBC the Legendre transform of the imaginary part is defined as

$$\Gamma_{\text{OBC}}^{\text{phot im}}[A^{\text{im}}] = \Im W_{\text{OBC}}^{\text{phot}}[j] - j \cdot A^{\text{im}}, \quad (40)$$

where

$$A^{\text{im}} = \frac{\delta \Im W_{\text{OBC}}^{\text{phot}}[j]}{\delta j}. \quad (41)$$

The corresponding equation of motion is

$$j = -\frac{\delta \Gamma_{\text{OBC}}^{\text{phot im}}[A^{\text{im}}]}{\delta A^{\text{im}}}. \quad (42)$$

The generating functional (28) yields the explicit form

$$\Gamma_{\text{OBC}}^{\text{phot im}}[A^{\text{im}}] = -\frac{1}{2} A^{\text{im}} \cdot \Im D_0^{-1} \cdot A^{\text{im}}. \quad (43)$$

For FBC the functional $\Im W[\hat{j}]$ depends on both currents and the Legendre transformation results in

$$\Gamma_{\text{FBC}}^{\text{phot im}}[A^{\text{im}}, A^{\text{adv im}}] = \Im W_{\text{FBC}}^{\text{phot}}[j, \bar{j}] - j \cdot A^{\text{im}} - \bar{j} \cdot A^{\text{adv im}} \quad (44)$$

where

$$\begin{aligned} A^{\text{im}} &= \frac{\delta \Re W_{\text{FBC}}^{\text{phot}}[j, \bar{j}]}{\delta j}, \\ A^{\text{adv im}} &= \frac{\delta \Re W_{\text{FBC}}^{\text{phot}}[j, \bar{j}]}{\delta \bar{j}}. \end{aligned} \quad (45)$$

The actual form,

$$\Gamma_{\text{FBC}}^{\text{phot im}}[A^{\text{im}}, A^{\text{adv im}}] = -\frac{1+\kappa^2}{4} A^{\text{adv im}} \Im D_0^{-1} A^{\text{adv im}} - A^{\text{im}} \Im D_0^{-1} A^{\text{im}} + \kappa A^{\text{im}} \Im D_0^{-1} A^{\text{adv im}}, \quad (46)$$

follows from the functional (29).

These effective actions will be used in Section VIA.

IV. CURRENT DYNAMICS IN THE NON-INTERACTING DIRAC-SEE

For a non-interacting system the determination of the expectation value of a local observable is trivial as long as the observable is a one-body operator, as in the case of the photon field discussed in the previous Section. The triviality comes from the fact that the generating functional for the connected Green function of the elementary fields is quadratic and can exactly be calculated. Higher order Green functions, $\langle 0|T[\phi_{x_1} \cdots \phi_{x_{2n}}]|0\rangle$ with $n \geq 2$, factorize to a sum of n disconnected products of two-point functions, according to Wick-theorem. The physical origin of this factorization is obvious, it comes from the absence of interactions between the particles. Each particle created by one of the operators must be destroyed by another operator in order to find a non-vanishing contribution to the vacuum expectation value. In other words, the largest cluster with non-factorizable structure contains two operators. But we meet serious difficulties as soon as the expectation value of such an operator is sought which controls more than one particle. The non-factorization arises from the possibility of creating particles by one operator which are annihilated by different other operators. One has to perform a "bosonisation" in a fermionic system because only bosonic operators can have non-vanishing expectation values. The simplest and most important bosonic operator is the electric current j considered in this Section.

Let us first introduce the generating functional, $W^{\text{el}}[\hat{a}]$, for the connected Green functions of the electric current by means of the functional

$$e^{iW^{\text{el}}[\hat{a}, \hat{\eta}, \hat{\eta}]} = \int D[\hat{\psi}] D[\hat{\bar{\psi}}] e^{i\hat{\bar{\psi}} \cdot [\hat{G}_0^{-1} + \hat{\mathcal{A}}] \cdot \hat{\psi} + i\hat{\eta} \cdot \hat{\bar{\psi}} + i\hat{\bar{\psi}} \cdot \hat{\eta} + iS_{CT}^e[\hat{a}]} \quad (47)$$

where the two-component fields $\hat{\psi} = (\psi_+, \psi_-)$ $\hat{a} = (a^+, a^-)$ were introduced together with the inverse electron propagator

$$\hat{G}_0^{-1} = \begin{pmatrix} G_0^{-1} & \\ 0 & -\gamma^0 G_0^{-1\dagger} \gamma^0 \end{pmatrix} + \hat{G}_{\text{BC}}^{-1}, \quad (48)$$

where $G_0^{-1} = i\partial - m$. We shall need later the composite operator counterterm

$$S_{CT}^e[\hat{a}] = \frac{\Delta Z_3 + \beta}{2} \hat{a} \cdot \begin{pmatrix} D_T^{-1} & 0 \\ 0 & -D_T^{-1*} \end{pmatrix} \cdot \hat{a} \quad (49)$$

with ΔZ_3 being UV divergent and the finite part β being fixed by a renormalization condition.

In the absence of charges in the initial and final states we identify the generating functional as $W_0^{\text{el}}[\hat{a}] = W^{\text{el}}[\hat{a}, 0, 0]$. When the system contains n^- electrons and n^+ positrons we take $\rho_i = |\Psi_i\rangle\langle\Psi_i|$ or $\rho_f = |\Psi_f\rangle\langle\Psi_f|$,

$$|\Psi\rangle = \prod_{j=1}^{n^-} \left(\int_{\mathbf{y}} \bar{\psi}_{t, \mathbf{y}} \chi_{j, \mathbf{y}}^- \right) \prod_{k=1}^{n^+} \left(\int_{\mathbf{y}} \bar{\chi}_{k, \mathbf{y}}^+ \psi_{t, \mathbf{y}} \right) |0\rangle, \quad (50)$$

with $t = t_i$ or t_f , respectively. The wave functions χ^- and $\bar{\chi}^+$ describe the one-particle e^\pm states. The generating functional for the current is then written as

$$e^{iW^{\text{el}}[\hat{a}]} = \prod_{\sigma=\pm 1} \prod_{j=1}^{n^-} \left(\int_{\mathbf{x}} \bar{\chi}_{j, \mathbf{x}}^- \frac{\delta}{\delta \bar{\eta}_{t_f, \mathbf{x}}^\sigma} \int_{\mathbf{y}} \chi_{j, \mathbf{y}}^- \frac{\delta}{\delta \eta_{t_i, \mathbf{y}}^\sigma} \right) \prod_{k=1}^{n^+} \left(\int_{\mathbf{x}} \chi_{k, \mathbf{x}}^+ \frac{\delta}{\delta \eta_{t_f, \mathbf{x}}^\sigma} \int_{\mathbf{y}} \bar{\chi}_{k, \mathbf{y}}^+ \frac{\delta}{\delta \bar{\eta}_{t_i, \mathbf{y}}^\sigma} \right) e^{iW^{\text{el}}[\hat{a}, \hat{\eta}, \hat{\eta}]}_{|\hat{\eta}=\hat{\eta}=0}. \quad (51)$$

We want to generate the expectation value of the current

$$j_x^\mu = \frac{1}{2} [\bar{\psi}_x \gamma^\mu \psi_x - (\gamma^\mu \psi_x)^{\text{tr}} \bar{\psi}_x^{\text{tr}}] \quad (52)$$

which changes sign under charge conjugation. To this end we use the γ -matrices

$$(\gamma_x^\mu)_{y, z} = \frac{1}{2} (\delta_{y, x+\eta e^\mu} \delta_{z, x-\eta e^\mu} + \delta_{y, x-\eta e^\mu} \delta_{z, x+\eta e^\mu}) \gamma^\mu, \quad (53)$$

where $\eta = 0^+$ in the Dirac Lagrangian and the minimal coupling. In this notation the current is $j_a = \bar{\psi} \cdot \gamma_a \cdot \psi$.

The mathematical source of the complications is that

$$W^{\text{el}}[\hat{a}] = -i \text{Tr} \ln \hat{G}^{-1}[\hat{a}] + S_{CT}[\hat{a}] \quad (54)$$

where $\hat{G}^{-1}[\hat{a}] = \hat{G}_0^{-1} + \hat{\mathcal{A}}$ is an involved functional even for the vacuum, ie. in the absence of initial or final charges. It is easy to understand the origin of this complication in the special case of $\rho_i = \rho_f = |0\rangle\langle 0|$ and time-independent source \hat{a} . The generating functional $W^{\text{el}}[\hat{a}]$ is now the difference of the energy of the Dirac-see in the presence of the vector potential a^+ and a^- . A local modification of the vector potentials creates a non-local polarization of the Dirac-see with all filled one-electron states contributing. Therefore, the infinitely many negative energy states filled up in the Dirac-see renders the functional highly non-trivial. It is not necessary that the states be filled. For scalar particles the structure of the generating functional remains the same except of the change of the overall sign. The external potential coupled to the particles creates a polarization of the ground state which involves an arbitrary number of (non-interacting) particles. The expectation value of bilinear operators measures the polarization created by the given boundary condition in time and its space-time dependence satisfies highly non-trivial equations reflecting the multi-particle dynamics of polarization in the ground state.

Our strategy followed in the case of the non-interacting Dirac-see will be similar to that of Section III, ie. we calculate first the generating functional for the connected Green functions of the current and then proceed with the construction of the effective actions. For weak fields the functionals can easily be calculated in the framework of the perturbation expansion, which is the main task of this Section. The issue of strong fields is far more difficult and will only be commented on briefly.

A. Connected Green functions

We start with the simpler case when there are no additional charges immersed in the Dirac-see. For weak external sources the functional (54) can be written as a functional Taylor series,

$$W^{\text{el}}[\hat{a}] = \sum_{n=1}^{\infty} \frac{1}{n!} W_0^{\text{el}}{}_{a_1, \dots, a_n} \hat{a}_{a_1} \cdots \hat{a}_{a_n} \quad (55)$$

where the super-index $a = (\pm, x, \mu)$ identifies a time axis, a space-time location and a vector index, and the coefficients give the connected Green functions. The expansion of the logarithmic function in Eq. (54) results in

$$W^{\text{el}}[\hat{a}] = -i \text{Tr} \ln \hat{G}_0^{-1} + i \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Tr}(\hat{G}_0 \cdot \hat{\not{a}})^n + S_{CT}[\hat{a}] \quad (56)$$

and

$$W_{a_1, \dots, a_n}^{\text{el}} = \frac{i(-1)^n}{n} \sum_{P \in S_n} \text{Tr}[\hat{G}_0 \cdot \gamma_{\hat{a}_{P(1)}}^s \cdot \hat{G}_0 \cdot \gamma_{\hat{a}_{P(2)}}^s \cdots \hat{G}_0 \cdot \gamma_{\hat{a}_{P(n)}}^s \cdot \hat{G}_0 \cdot \gamma_{\hat{a}}^s] - \delta_{n,2}(\Delta Z_3 + \beta) \hat{D}_T \quad (57)$$

where

$$\gamma_x^a = \begin{pmatrix} \gamma_x^a & 0 \\ 0 & \gamma_x^a \end{pmatrix}. \quad (58)$$

The symmetrization with respect to the exchange of the external legs is achieved by the summation over the permutations of the vertices in (57). The odd orders are vanishing according to Furry's theorem. For the sake of simplicity we truncate the generating functional at the quadratic order and write

$$W^{\text{el}}[\hat{a}] = -\frac{1}{2} \hat{a} \cdot \hat{G}_R \cdot \hat{a} \quad (59)$$

where the renormalized current two-point function $\hat{G}_R = \hat{G}_0 - \beta \hat{D}_T$ is given by

$$\begin{aligned} \tilde{G}_{0a,b}^{++} &= -i \text{Tr}[\hat{G}_0^{++} \cdot \gamma_a \cdot \hat{G}_0^{++} \cdot \gamma_b] + \Delta Z_3 T(\square - i\epsilon) \\ \tilde{G}_{0a,b}^{+-} &= -i \text{Tr}[\hat{G}_0^{+-} \cdot \gamma_a \cdot \hat{G}_0^{+-} \cdot \gamma_b] \\ \tilde{G}_{0a,b}^{--} &= -i \text{Tr}[\hat{G}_0^{--} \cdot \gamma_a \cdot \hat{G}_0^{--} \cdot \gamma_b] - \Delta Z_3 T(\square + i\epsilon). \end{aligned} \quad (60)$$

The notation $\tilde{G} = \tilde{G}^{++}$ will be used in the rest of the paper. Well known results (eg. Ref. [47]) include the renormalized two point function

$$\begin{aligned} \tilde{G}_{0q}^{\mu\nu} &= \int_x e^{-iqx} \tilde{G}_{x,0}^{\mu\nu} \\ &= T^{\mu\nu} \left[\frac{1}{15\pi} \frac{q^4}{m^2} + O\left(\frac{q^6}{m^4}\right) \right] \end{aligned} \quad (61)$$

obtained in the framework of the gradient expansion where $\Im \tilde{G} = 0$ since the creation of a mass-shell $e^- e^+$ pair is forbidden. We shall use the same parameterization of the propagator

$$\hat{G}_0 = \begin{pmatrix} \tilde{G}_0^{\text{near}} + i\Im \tilde{G}_0 & -\frac{1}{2} \tilde{G}_0^{\text{far}} + i\Im \tilde{G}_0 \\ \frac{1}{2} \tilde{G}_0^{\text{far}} + i\Im \tilde{G}_0 & -\tilde{G}_0^{\text{near}} + i\Im \tilde{G}_0 \end{pmatrix}. \quad (62)$$

as for photons and the retarded and advanced current Green functions will be defined by $\tilde{G}_0^{\text{ret}} = \tilde{G}_0^{\text{near}} + \frac{1}{2} \tilde{G}_0^{\text{far}}$ and $\tilde{G}_0^{\text{adv}} = \tilde{G}_0^{\text{near}} - \frac{1}{2} \tilde{G}_0^{\text{far}}$, respectively.

An interesting feature of Eq. (57) is the need of renormalization. The Green functions $W_0^{\text{el}}{}_{a_1, \dots, a_n}$ are finite for $n \geq 3$ but the two-point function diverges. In fact, this two-point function is identical to the one-loop photon self energy except of the missing factor e^2 . The electrons of the Dirac-see tend to approach each others too frequently and make the two point function divergent when the two legs approach each other in space-time. This is a well

known problem in QED and the cure is the introduction of the counterterm $S_{CT}[\hat{a}]$. The UV finite part, β , of the counterterm is fixed by a renormalization condition to be imposed. The lesson of this divergence, a relativistic effect, is that the dynamics of the current j can not be defined without an additional scale, the cutoff, even in the absence of interactions.

The need of a renormalization condition for non-interacting particles demonstrates a characteristic difference between first and second quantized systems. In quantum mechanics observables are defined by the operators. In quantum field theory however the observables represented by composite operators may need counterterms and their proper definition must include the corresponding renormalization condition. Symmetry principles can not fix the counterterm to the current as long as it is transverse and we find a one-parameter family of current with the two-point function

$$i\langle 0T[j_x^\mu j_{x'}^\nu]0\rangle + \beta\Box\delta_{x,x'}T^{\mu\nu}, \quad (63)$$

as far as the many-body aspects are concerned. The finite part of the counterterm influences the product of two current operators at zero separation and this contact term plays an important role even at finite energies. We shall see later that the current coupled to the photon field is defined by $\beta = 0$.

The generating functional is more complicated in the presence of additional charges in the Dirac-see. We then have the additional term $\hat{\eta} \cdot \hat{G}[\hat{a}]_0 \cdot \hat{\eta}$ in $W[\hat{a}, \hat{\eta}, \hat{\eta}]$ which gives for instance

$$W^{\text{el}}[\hat{a}] = W_{\text{FBC}}^e[\hat{a}] - i \sum_{\sigma=\pm} \ln(\hat{G}[\hat{a}]_{(j_f, x_f), (j_i, x_i)}^{\sigma\sigma}) \quad (64)$$

for charged fixed boundary condition with a single electron. The initial (final) state is characterized by the space-time point x_i (x_f) with bispinor index j_i (j_f) and $a = (x, j)$ denotes the combined index. As an other example, the open boundary condition for two electrons with a_n , $n = 1, 2$ in the initial state yields

$$W^{\text{el}}[\hat{a}] = W_{\text{OBC}}^e[\hat{a}] - i \ln(\hat{G}[\hat{a}]_{a_1 a_1}^{+-} \hat{G}[\hat{a}]_{a_2 a_2}^{+-} - \hat{G}[\hat{a}]_{a_1 a_2}^{+-} \hat{G}[\hat{a}]_{a_2 a_1}^{+-}). \quad (65)$$

Now the generating functional contains odd orders in \hat{a} . The different structure of the see and valence contributions, the first and the second terms on the right hand side of Eqs. (64)-(65), reflects the fact that the Dirac-see is made up of negative energy one-particle states while the additional charges, introduced by the creation operators, correspond to positive energy. As a result, the valence charges move freely while the motion of the particles making up the Dirac-see is restricted by the Pauli-blocking.

B. Effective actions

The equation of motion for the current expectation values is derived from the effective action

$$\Gamma^{\text{el}}[J, J^{\text{adv}}] = \Re W^{\text{el}}[a, \bar{a}] - \bar{a} \cdot J^{\text{adv}} - a \cdot J \quad (66)$$

involving the sources

$$a^\pm = a \frac{1 \pm \kappa}{2} \pm \bar{a} \quad (67)$$

and independent variables

$$\begin{aligned} J &= \frac{\delta \Re W^{\text{el}}[a, \bar{a}]}{\delta a}, \\ J^{\text{adv}} &= \frac{\delta \Re W^{\text{el}}[a, \bar{a}]}{\delta \bar{a}} \end{aligned} \quad (68)$$

which are conserved currents.

The effective action for the physical field A only is defined by

$$\Gamma^{\text{el}}[J] = \Re W^{\text{el}}[a, \bar{a}] - a \cdot J. \quad (69)$$

The actual expressions for the Dirac-see can be obtained by the replacements $D_0 \rightarrow \tilde{G}$, $A \rightarrow J$, and $A^{\text{adv}} \rightarrow J^{\text{adv}}$ in Eqs. (34), (37) and (39).

When charges are added to the Dirac-see the valence propagator contributes to the \hat{a} -dependence in $W^{\text{el}}[\hat{a}]$. This requires $\hat{j} \neq 0$ and a redefinition of \hat{G} in Eq. (59) to be used in the Legendre transformation.

C. Polarized charges

Let us consider now the dynamics of localized charges polarized from the Dirac-see and described by the effective action truncated at the quadratic order,

$$\Gamma^{\text{el}}[J] = \frac{1}{2\kappa} J \cdot \Gamma^{(2)\text{el}} \cdot J. \quad (70)$$

The current is supposed to be slowly varying over distances $\ell < 1/m$ rendering the leading order gradient expansion (61)

$$\Gamma^{(2)\text{el}} = -\frac{T}{\frac{1}{15\pi m^2}(\Box - i\epsilon)^2 - \beta(\Box - i\epsilon)} \quad (71)$$

applicable. Feynman's ϵ -prescription is displayed in the expression explicitly for the calculation of the inverse of the kernel. This expression obtained for OBC holds for FBC, as well, as long as the total charge is vanishing and J^μ consists of closed flux-tubes. The calculation indicated briefly in Appendix B for the choice $\beta = 0$, justified for the electric current in Section V A, gives

$$\Gamma_{x,x'}^{(2)\text{el}\mu\nu} = -\frac{15m^2}{8}\Theta((x-x')^2)T^{\mu\nu}, \quad (72)$$

ie. the vertex function is step function, assuming the value 0 or $-15m^2/8$ for spacelike or timelike separations, respectively.

Let us now assume the form

$$J_x^\mu = g^{\mu 0} \sum_j N_j \rho(|\mathbf{x} - \mathbf{x}_j|) \quad (73)$$

for the current where N_j denotes the number of electrons of a localized state and $\int_{\mathbf{x}} \rho(r) = 1$. Notice that the charge is polarized out from the negative energy one-particle states for weak external source and there is no reason for N_j to be integer. The average distance between two world-tubes is given by

$$r_{jk} = \int_{\mathbf{y}, \mathbf{z}} \rho(|\mathbf{y} - \mathbf{x}_j|) |\mathbf{y} - \mathbf{z}| \rho(|\mathbf{z} - \mathbf{x}_k|). \quad (74)$$

To identify the total mechanical energy we need $E_{\text{tot}} = -\Gamma_0^{\text{el}} \text{OBC}[J]/(t_f - t_i)$ in the limit $t_f - t_i \rightarrow \infty$,

$$E_{\text{tot}} = \frac{15m^2}{8} \left[-(t_f - t_i) N_{\text{tot}}^2 + \sum_{j,k} N_j N_k r_{jk} \right] \quad (75)$$

$N_{\text{tot}} = \sum_j N_j$ being the total charge. The effective action for $\beta = 0$ is IR finite for neutral systems only. This is a trivial result, reflecting the impossibility of polarizing out a net charge by a neutral external source.

It is interesting to note that the action of the current defined by $\beta < 0$ is that of a classical electrodynamical action with charge $g = 1/\sqrt{-\beta}$ and electric susceptibility $O(g^2 k^2/m^2)$, cf. Section VI C yielding

$$E_{\text{tot}} = -\frac{g^2}{8\pi} \sum_{j,k} N_j N_k \int_{\mathbf{x}, \mathbf{y}} \rho(|\mathbf{x} - \mathbf{x}_j|) \frac{1 - e^{-\sqrt{15\pi}|\mathbf{x} - \mathbf{y}|mg}}{|\mathbf{x} - \mathbf{y}|} \rho(|\mathbf{y} - \mathbf{x}_k|). \quad (76)$$

The screening effect of vacuum polarization detected by the current defined with $\beta < 0$ removes the IR divergence and allows a net charge.

Though it is perplexing to find interactions between localized charges in the non-interacting Dirac-see there is a simple explanation for it. In fact, let us consider the normalized two-particle state

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2N}} [\psi_1(\mathbf{x}_1) \psi_2(\mathbf{x}_2) - \psi_1(\mathbf{x}_2) \psi_2(\mathbf{x}_1)] \quad (77)$$

in non-relativistic Quantum Mechanics where $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ are two not necessarily orthogonal wave functions. The matrix element

$$\langle \psi | \Delta | \psi \rangle = \frac{1}{N} [\langle \psi_1 | \Delta | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle + \langle \psi_2 | \Delta | \psi_2 \rangle \langle \psi_1 | \psi_1 \rangle - 2 \text{Re} \langle \psi_1 | \Delta | \psi_2 \rangle \langle \psi_2 | \psi_1 \rangle] \quad (78)$$

shows that the kinetic energy receives a contribution from the entanglement of the two particle states in the region where the one-particle wave functions overlap and this additional piece in the energy might be interpreted as an interaction potential

$$U(\mathbf{x}_1, \mathbf{x}_2) = -\frac{\text{Re}\langle\psi_1|\Delta|\psi_2\rangle\langle\psi_2|\psi_1\rangle}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle - |\langle\psi_1|\psi_2\rangle|^2}. \quad (79)$$

All this is the well known Pauli-blocking: the anti-symmetrization excludes certain kinematical regions leading thereby to an increase of the kinetic energy. This blocking is restricted to small separations since the one-particle states must overlap.

But this picture is static, ignoring the dynamics. The modification of the density distribution can be viewed as a result of polarizations due to particle-anti particle pairs whose propagation establishes non-local exchange correlation effects during some time. The near-field vertex function obtained from the $O(\square^{-1})$ kernel of the non-physical current with $\beta < 0$ restricts correlations onto the light-cone. This is what happens in QED, namely the mixing of the states of the photon and the electron-positron pair in a gauge and Lorentz invariant dynamics keeps the polarization of the Dirac-see on the mass shell $m^2 = 0$. But the the UV regime of the $O(\square^{-2})$ kernel of the true electric current is suppressed and the singularity at the light-cone, ie. at vanishing invariant length is reduced to a finite discontinuity leaving no asymptotic particle-like states contributing to the current two-point function. It is remarkable that the vertex function is actually a step function and it establishes a distance independent force between static world-tubes in sufficiently long time.

Notice furthermore that it is not necessary to have relativistic fermions to find such a long-range exchange correlation. Charged relativistic bosons display a similar dynamics, the only difference compared to the fermion case being an overall sign in the generating functional $W[\hat{a}]$ apart of the modification of the current operator the external field \hat{a} is coupled to. But both the bosonic and the fermionic currents are bilinears of the elementary fields and, therefore, the vacuum polarization should be qualitatively similar. It is not the filled states of the Dirac-see which are important but rather the propagation of states created by the current operator from the vacuum. These excitations are always bosonic and thus subject to an effective bosonic description. Non-relativistic systems, such as non-interacting fermions at finite density display similar effect, too.

This distance-independent force would not be observable even if one had sufficient space resolution in the experimental device. The linear potential is a special feature of the non-interacting Dirac-see only and becomes screened by the electromagnetic interactions. In fact, suppose we had a matter and an anti-matter localized states and we started to separate the two components. Increasing the separation beyond the Compton wavelength of the electron costs energy comparable with those needed to put a virtual e^-e^+ pair on the mass-shell. Furthermore, the gradient expansion is valid for separations larger than the Compton wavelength. Therefore the charge of the real particles which are created by the minimal coupling vertex tends to screen the original charges and only localized states with non-integer charge remain strongly correlated.

D. Localized states

The preceding discussion of the dynamics of the polarized current is restricted to weak external sources. The involved, non-polynomial character of the generating functional $W_0^{\text{el}}[\hat{a}]$ makes its appearance for strong external sources. Though the correspondence $\hat{a} \rightarrow \hat{J}$ is unique but its inverse is not and the effective actions are not uniquely defined for strong external sources. By retaining the profile of the current only one loses important information. For instance, a given charge density has different dynamics depending on the nature of the states which contribute to the charge. This is reminiscent of the small and large polaron problem in solid state physics where the same polarization may induce different dynamical responses depending on the way the polarization builds up.

For weak external sources the polarized charge is made up by small contributions received from a number of negative energy one-particle states. The resulting charge density can be localized but extended states contribute only. As a result, Pauli-blocking renders the dynamics of such localized states highly correlated. For slightly stronger localized external field a bound states are formed in the mass gap. This case will be considered in a qualitative manner below. For stronger localized external field the localized states may appear in the negative energy continuum. Such bound states appear as a violation of the convexity of the effective action; a rearrangement of the vacuum takes place. One member of a virtual e^+e^- pair created by the external field jumps into the potential well and the resulting energy is sufficient to put the other charge on mass-shell. As a result, a new, well defined Legendre transform of $W_0^e[\hat{a}]$ is regained. In other words, the effective action, $\Gamma^{\text{el}}[\hat{J}]$, is multi-valued and may have several consistent "sheets" for a given current.

Let us return to the case of medium strong, localized external field which create localized states in the mass gap. The external fields $\hat{\eta}$ and $\hat{\bar{\eta}}$ can be used to place charges into the initial or final state which can be captured by these

potential centers. These charges have weak overlap with the filled extended states and are supposed to obey the "free dynamics" anticipated from classical mechanics. We start with charges of the same sign with fixed boundary condition where $W^{\text{el}}[\hat{a}] = W[\hat{a}]_0^c - i \ln G[\hat{a}]$, $G[\hat{a}]$ being the valence propagator with given initial and final points. We are interested in the possible decoupling of the trajectories. Therefore, the initial and final location and spin of the charges are chosen to be the same. The external potential is chosen to be static and of the form $a_\mu = g_{\mu 0} u$ where the temporal component, u , is the sum of potential wells which are strong enough to create localized state in the mass gap and spread enough such that the characteristic size of the bound state, ℓ , be large compared to the Compton wavelength of electrons, $m\ell > 1$, in order to avoid pair creation. We choose $t_f - t_i \gg 1/m$ and shall apply the non-relativistic approximation for the valence propagator. What is crucial is that $W[\hat{a}]_0^c$ can then be ignored in $W^{\text{el}}[\hat{a}]$ beside of the valence propagator. Once the vacuum polarization is suppressed, the rest is a simple problem in non-relativistic quantum mechanics. In fact, we have at this point

$$\Gamma_{\text{CFBC}}^{\text{el}}[J] = -(t_f - t_i)(m + E_0) - \int_{t_i}^{t_f} dt \int d^3x u(\mathbf{x}) J^0(t, \mathbf{x}) \quad (80)$$

for $t_f - t_i \rightarrow \infty$ where E_0 denotes the ground state energy of electrons bound to the potential wells. When separating the wells the charge density remains localized at the wells and the contribution of the last term on the right hand side breaks off into a sum of the contributions of separate wells. The correlation energy of Eq. (79) becomes small for well separated states and therefore the contributions of the potential wells to E_0 decouple.

Once the separation of localized charges is established we can consider the issue of the dynamics of a single localized state. Let us suppose that the current J is a narrow flux tube with flux N and its center follows the world-line $x^\mu(s)$ in space-time. One interprets this as the motion of a charge Ne along the world line $x^\mu(s)$. If the value of the effective action evaluated for this current, divided by the invariant length of the world line $x^\mu(s)$ is independent of the time of evolution then this ratio is the mechanical mass. Let us consider a world line without acceleration. Then Lorentz invariance allows us to set $\mathbf{x}_i = \mathbf{x}_f$. We use the same external potential as in the previous case except that it has a single potential well only and we place a single electron in the initial state. The generating functional for this problem is (64) whose value can be written as in Eq. (80). The ground state energy, E_0 , is the sum of the expectation values of the kinetic and the potential energies. The latter cancels against the last term on the right hand side of Eq. (80) because $J^0(t, \mathbf{x})$ is the particle density, ie. the magnitude square of the ground state wave function apart of some finite time at the beginning and at the end of the motion which give negligible contribution in the limit $t_i = t_f \rightarrow \infty$. We are finally left with $\Gamma^{\text{el}}[J] = -(t_f - t_i)(m + E_k)$ and the mechanical mass, the coefficient in front of the first term on the right hand side of Eq. (1), can be written as

$$M = m + \Delta m_k + \Delta m_{\text{pol}} + \Delta m_\gamma, \quad (81)$$

where Δm_k is the expectation value of the kinetic energy in the bound state which is approximately $1/(\ell^2 m)$. The second term represents the effects of the polarizations of the Dirac-see by the external potential coming from $W[\hat{a}]_0^c$. Its order of magnitude, $m\ell u$, is estimated as the product of the polarized charge square, assumed to be proportional to u/m , the ratio of the energy scale of the potential and the mass, the length scale of the potential well ℓ and the string tension $\sigma \approx m^2$ introduced in Section IV C. Finally, Δm_γ stands for the radiative corrections arising from the electromagnetic interaction and is e^2/ℓ .

The mechanical mass depends on the details the elementary particles localized in their world-tubes. Once the coupling to the photon field is introduced the point-like charges develop their vacuum polarization cloud, $\ell \approx 1/m$, and the correction to the mechanical mass becomes non-perturbative, $O(m)$, but uniquely defined and can be removed by renormalization of the mass.

V. INTERACTING ELECTRONS AND PHOTONS

After the separate discussion of the dynamics of the expectation value of the photon field and the electron current we embark the case of the interacting system.

A. Generating functional for connected Green functions

The generating functional for the connected Green functions of the current and the photon field is

$$e^{iW[\hat{a}, \hat{j}]} = \int D[\hat{\psi}] D[\hat{\bar{\psi}}] D[\hat{A}] e^{i\hat{\bar{\psi}} \cdot [\hat{G}_0^{-1} + \hat{\sigma}(\hat{\mathcal{A}} - e\hat{A})] \cdot \hat{\psi} + \frac{i}{2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} + i\hat{j} \hat{\sigma} \cdot \hat{A} + iS_{CT}} \quad (82)$$

where the minimal coupling contains the matrix

$$\hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (83)$$

The perturbative renormalization is carried out by using the counterterms $S_{CT} = S_{CT}^{QED} + S_{CT}^{e\gamma}$ where S_{CT}^{QED} contains the usual counterterm needed for the renormalization of the electron and photon Green functions in a given order of the loop-expansion. The additional term is chosen to be

$$S_{CT}^{e\gamma} = -(\Delta Z_3 - \alpha)e\hat{a} \cdot \hat{D}_0^{-1} \cdot \hat{A} + \frac{1}{2}(\Delta Z_3 + \beta)\hat{a} \cdot \hat{D}_0^{-1} \cdot \hat{a} \quad (84)$$

The origin of this expression becomes clear by considering the photon part of the action, the sum of $S_{CT}^{e\gamma}$, the bare action and the photon self energy counterterm,

$$S_{CT}^{e\gamma} + \frac{e^2}{2e_B^2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} = \frac{1}{2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} + \frac{\Delta Z_3}{2} (e\hat{A} - \hat{a}) \cdot \hat{D}_0^{-1} \cdot (e\hat{A} - \hat{a}) + \alpha \hat{a} \cdot \hat{D}_0^{-1} \cdot \hat{A} + \frac{\beta}{2} \hat{a} \cdot \hat{D}_0^{-1} \cdot \hat{a}, \quad (85)$$

by using

$$\frac{1}{e_B^2} = \frac{1}{e^2} + \Delta Z_3. \quad (86)$$

The second term on the right hand side with $\hat{a} = 0$ is the counterterm for the photon self energy but its dynamical origin is in the fermionic sector of the theory. Any vector potential which couples to the electric current as the photon field must appear in this counterterm on equal footing with A . This shows that the choice $\alpha = \beta = 0$ of the finite part of the counterterms is needed to identify the current coupled to the source a with the electromagnetic current.

The electron field can be integrated out easily in Eq. (82), leaving behind

$$e^{iW[\hat{a}, \hat{j}]} = \int D[\hat{A}] e^{iW^{\text{el}}[\hat{a} - e\hat{\sigma}\hat{A}] + \frac{i}{2}\hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} + i\hat{j} \cdot \hat{A} + iS_{CT}} \quad (87)$$

where $W^a[\hat{a}]$ stands for the generator functional of the connected Green functions of the current in the non-interacting Dirac-see as given in eqs. (54), (64) or (65) for different boundary conditions. The integration over the photon field can be carried out in the loop-expansion, see Appendix C for the details. We record here the results for the quadratic part of the generating functional in the absence of charges in the initial state,

$$W[\hat{a}, \hat{j}] = -\frac{1}{2}(\hat{a}, \hat{j}) \cdot \begin{pmatrix} \hat{\hat{G}} & e\hat{\hat{G}} \cdot \hat{\sigma} \hat{D}_0 \\ e\hat{D}_0 \hat{\sigma} \cdot \hat{\hat{G}} & \hat{D} \end{pmatrix} \cdot \begin{pmatrix} \hat{a} \\ \hat{j} \end{pmatrix} \quad (88)$$

where

$$\hat{\hat{G}} = \frac{1}{\hat{\hat{G}}_0^{-1} - \hat{\Sigma}^{\text{el}}} \quad (89)$$

is the full current two-point function containing the current self-energy $\hat{\Sigma}^{\text{el}}$ and

$$\hat{D} = \frac{1}{\hat{D}_0^{-1} - \hat{\Sigma}^{\text{phot}}} \quad (90)$$

stands for the photon propagator involving the photon self-energy $\hat{\Sigma}^{\text{phot}}$. The formal expressions for the self-energies up to two-loops are

$$\hat{\Sigma}^{\text{el}} = e^2 \hat{\sigma} \hat{D}_0 \hat{\sigma} - e^2 \hat{\hat{G}}_0^{-1} \cdot \hat{W}_D^{\text{el}(4)} \cdot \hat{\hat{G}}_0^{-1} \quad (91)$$

and

$$\hat{\Sigma}^{\text{phot}} = e^2 \hat{\sigma} (\hat{\hat{G}}_0 - e^2 \hat{W}_D^{\text{el}(4)}) \hat{\sigma} \quad (92)$$

where the first and second terms on the right hand sides represent the one- and two-loop corrections with

$$W_{Dab}^{\text{el}(4)} = W_{abcd}^{\text{el}(4)} (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{cd} \quad (93)$$

being the one-loop self-energy and vertex correction to the current two-point function.

The more useful parameterizations (14), (67) yields the form

$$\Re W[a, \bar{a}, j, \bar{j}] = \frac{1}{2}(a, j, \bar{a}, \bar{j}) \cdot \begin{pmatrix} \frac{\kappa}{2}(K^{\text{tr}} + K) & K \\ K^{\text{tr}} & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ j \\ \bar{a} \\ \bar{j} \end{pmatrix}, \quad (94)$$

with

$$K_{\text{OBC}} = - \begin{pmatrix} \tilde{G}^{\text{ret}} & e\tilde{G}^{\text{ret}} \cdot D_0^{\text{ret}} \\ eD_0^{\text{ret}} \cdot \tilde{G}^{\text{ret}} & D^{\text{ret}} \end{pmatrix}, \quad (95)$$

and

$$K_{\text{FBC}} = - \begin{pmatrix} \tilde{G}^{\text{near}} & e(\tilde{G}^{\text{near}} \cdot D_0^{\text{near}} - \Im \tilde{G} \cdot \Im D_0) \\ e(D_0^{\text{near}} \cdot \tilde{G}^{\text{near}} - \Im D_0 \cdot \Im \tilde{G}) & D_0 \end{pmatrix}, \quad (96)$$

in the absence of charges with positive energy, for open and fixed boundary conditions, respectively. As pointed out after Eq. (36) the real part of the non-interacting generating functional for fixed boundary condition can be obtained from that of the open boundary condition by replacing the retarded and advanced Green functions by the near field version. Such a change is not sufficient for interacting system where higher loop may bring in the product of two imaginary parts into $\Re W$. However by restricting our discussion to distance scales beyond the Compton wavelength of the electron the creations of mass-shell e^+e^- pairs are suppressed and $\Im \tilde{G} = 0$, cf. Eq. (61) obtained in the framework of the gradient expansion. It is pointed out in Appendix D 2 that this is the only violation of the simple rule mentioned above and once the pair creation is excluded one can again obtain easily the fixed boundary condition expressions from their open boundary counterparts.

The comparison of Eqs. (C20) and (C21) reveals that the imaginary part of the propagator undergoes more substantial changes than the real part when the boundary condition is modified.

B. Legendre transformation

We introduce a number of effective actions. The simplest effective action arises from the Legendre transformation of the variables a^\pm and j^\pm ,

$$\Gamma[\hat{J}, \hat{A}] = W[\hat{a}, \hat{j}] - \hat{a} \cdot \hat{J} - \hat{j} \cdot \hat{A} \quad (97)$$

where

$$\hat{J} = \frac{\delta W}{\delta \hat{a}} \quad (98)$$

and

$$\hat{A} = \frac{\delta W}{\delta \hat{j}}. \quad (99)$$

The inverse of the block-matrix needed for the change of variables in the generating functional (88) can be read off from the result of Eq. (D3) yielding

$$\Gamma[\hat{J}, \hat{A}] = \Gamma^{\text{mech}}[\hat{J}] + \frac{1}{2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} - e \hat{A} \hat{\sigma} \cdot \hat{J} \quad (100)$$

where the first term is responsible for the dynamics of charges moving within the interacting Dirac-see and is given by

$$\Gamma^{\text{mech}}[\hat{J}] = \frac{1}{2} \hat{J} \cdot (\hat{G}_0^{-1} - \hat{G}_0^{-1} \cdot \hat{W}_D^{\text{el}(4)} \cdot \hat{G}_0^{-1}) \cdot \hat{J} \quad (101)$$

on the two-loop level. The equations of motions are

$$\frac{\delta \Gamma[\hat{J}, \hat{A}]}{\delta \hat{J}} = -\hat{a} \quad (102)$$

and

$$\frac{\delta\Gamma[\hat{J}, \hat{A}]}{\delta\hat{A}} = -\hat{j}. \quad (103)$$

What is remarkable in the expression (100) is that the electromagnetic interactions are represented without loop-corrections, ie. by the inverse of the free photon propagator and the minimal coupling without a form factor. This is because the introduction of separate variables for the current and the photon field formally removes the one-loop contributions from the action. In fact, the elimination of one of the fields by its equation of motion reintroduces the one-loop pieces in form of an action at a distance. In fact, eliminating \hat{J} or \hat{A} by their equations of motion generates the action at a distance effective actions

$$\Gamma[\hat{A}] = \frac{1}{2} \hat{A} \cdot \hat{D}^{-1} \cdot \hat{A} \quad (104)$$

or

$$\Gamma[\hat{J}] = \frac{1}{2} \hat{J} \cdot \hat{G}^{-1} \cdot \hat{J} \quad (105)$$

for $\hat{j} = \hat{a} = 0$, involving the full propagators.

Although the effective action (100) shows obvious similarities with the action of classical electrodynamics it is not satisfactory. In fact, though either fields of the type $+$ or $-$ can be used to read off measured expectation values but we have both of them in the effective action coupled to each other. The physical expectation values are easier to read off when the Legendre transformation is performed on the variables introduced by Eqs. (14) and (67). For this end we shall use the effective action

$$\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}] = \Re W[a, \bar{a}, j, \bar{j}] - J \cdot a - J^{\text{adv}} \cdot \bar{a} - A \cdot j - A^{\text{adv}} \cdot \bar{j} \quad (106)$$

with the new variables

$$J = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta a}, \quad J^{\text{adv}} = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta \bar{a}}, \quad A = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta j}, \quad A^{\text{adv}} = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta \bar{j}}. \quad (107)$$

The inverse Legendre transformation is based on the relations

$$a = -\frac{\delta \Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]}{\delta J}, \quad \bar{a} = -\frac{\delta \Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]}{\delta J^{\text{adv}}} \quad (108)$$

and

$$j = -\frac{\delta \Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]}{\delta A}, \quad \bar{j} = -\frac{\delta \Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]}{\delta A^{\text{adv}}}. \quad (109)$$

The block-matrix matrix of Eq. (95) yields, after some straightforward calculation outlined in Appendix D3, the effective action

$$\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}] = -\frac{1}{2} (J, A, J^{\text{adv}}, A^{\text{adv}}) \cdot \begin{pmatrix} 0 & K^{\text{tr}-1} \\ K^{-1} & -\frac{\kappa}{2} (K^{\text{tr}-1} + K^{-1}) \end{pmatrix} \cdot \begin{pmatrix} J \\ A \\ J^{\text{adv}} \\ A^{\text{adv}} \end{pmatrix}, \quad (110)$$

where

$$K^{-1} = - \begin{pmatrix} \tilde{G}_0^{\text{ret}-1} + e^2 \tilde{G}_0^{\text{ret}-1} \cdot W_D^{\text{el}(4)\text{ret}} \cdot \tilde{G}_0^{\text{ret}-1} & -e \\ -e & D_0^{\text{ret}-1} \end{pmatrix} \quad (111)$$

for open boundary condition. The retarded part $W_D^{\text{el}(4)\text{ret}} = \Re W_D^{\text{el}(4)++} + \Re W_D^{\text{el}(4)-+}$ is defined in the same manner as for the free block-propagator in Eq. (26), cf. Appendix D2. The equations of motion (D23) give $J^{\text{adv}} = A^{\text{adv}} = 0$ for vanishing off-shell sources, $a = j = 0$, as expected, and we find the equations of motion

$$\begin{aligned} J &= \frac{1}{1 + e^2 W_D^{\text{el}(4)r} \cdot \tilde{G}_0^{\text{ret}-1}} \cdot \tilde{G}_0^{\text{ret}} \cdot (eA - \bar{a}), \\ A &= D_0^{\text{ret}} \cdot (eJ - \bar{j}) \end{aligned} \quad (112)$$

for the physical fields. The one-loop contributions are again absent and the two-loop term represents the electromagnetic interaction in the Dirac-see in the first line of Eqs. (112) as in the effective action (100). According to the second line of Eqs. (112) both the external and the dynamical currents, \bar{j} and J induce retarded potentials.

In the case of fixed boundary condition we replace the retarded and advanced propagators by the near field version in the absence of creation of mass shell e^+e^- pairs and use

$$K^{-1} = - \begin{pmatrix} \tilde{G}_0^{\text{near}-1} + e^2 \tilde{G}_0^{\text{near}-1} \cdot W_D^{\text{el}(4)\text{near}} \cdot \tilde{G}_0^{\text{near}-1} & -e \\ -e & D_0^{\text{near}-1} \end{pmatrix} \quad (113)$$

in Eq. (110). The equations of motion become

$$\begin{aligned} \tilde{G}_0^{\text{near}} \bar{a} &= -(1 + e^2 W_D^{\text{el}(4)\text{near}} \cdot \tilde{G}_0^{\text{near}-1}) \cdot J + e \tilde{G}_0^{\text{near}} \cdot A, \\ D_0^{\text{near}} \bar{j} &= e D_0^{\text{near}} J - A. \end{aligned} \quad (114)$$

The simplest effective action includes physical fields only and it is defined as

$$\Gamma[J, A] = \Re W[a, \bar{a}, j, \bar{j}] - J \cdot a - A \cdot j \quad (115)$$

with the new variables

$$J = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta a}, \quad A = \frac{\delta \Re W[\hat{a}, \hat{j}]}{\delta j}, \quad (116)$$

and the source \bar{a} and \bar{j} treated as passive parameters in the Legendre transformation. Notice that the important block in the generating functional (94) we have to invert to find the equations of motion for the physical field, the upper left one, is the same for both boundary conditions. The dependence on the boundary condition appears through the linear pieces of the effective action only. The simple steps shown in Appendix D 4 lead to the effective action

$$\begin{aligned} \kappa \Gamma[J, A] &= \Gamma^{\text{mech}}[J] + \frac{1}{2} A \cdot D_0^{\text{near}-1} \cdot A - e A \cdot J - A \cdot J^{\text{ext}}, \\ \Gamma^{\text{mech}}[\hat{J}] &= \frac{1}{2} J \cdot (\tilde{G}_0^{\text{near}-1} + \tilde{G}_0^{\text{near}-1} \cdot W_D^{\text{el}(4)\text{near}} \cdot \tilde{G}_0^{\text{near}-1}) \cdot J - J \cdot A^{\text{ext}} \end{aligned} \quad (117)$$

where the external sources

$$\begin{aligned} J^{\text{ext}} &= D_0^{\text{near}-1} \cdot W_j - e W_a \\ A^{\text{ext}} &= (\tilde{G}_0^{\text{near}-1} + \tilde{G}_0^{\text{near}-1} \cdot W_D^{\text{el}(4)\text{near}} \cdot \tilde{G}_0^{\text{near}-1}) \cdot W_a - e W_j \end{aligned} \quad (118)$$

are given in terms of the boundary condition dependent terms W_a , W_j shown in Eqs. (D25) and (D26). The corresponding equations of motion for the field J is

$$J = -\tilde{G}_0^{\text{ret}}(\bar{a} + e D_0^{\text{ret}} \cdot \bar{j}) + \frac{e}{\tilde{G}_0^{\text{near}-1} + \tilde{G}_0^{\text{near}-1} \cdot W_D^{(4)n} \cdot \tilde{G}_0^{\text{near}-1}} [A + D_0^{\text{ret}} \cdot \bar{j} + e D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} \cdot \bar{a}] \quad (119)$$

where the upper (lower) indices correspond to open (fixed) boundary condition. The equation of motion for the electromagnetic field is

$$A = e D_0^{\text{near}} \cdot J - D_0^{\text{ret}} \cdot \bar{j} + \frac{e}{2} D_0^{\text{far}} \cdot \tilde{G}_0^{\text{ret}} \cdot (\bar{a} + e D_0^{\text{ret}} \cdot \bar{j}) + (e^4 D_0^{\text{ret}} \cdot W_D^{\text{el}(4)\text{ret}} \cdot D_0^{\text{ret}} - e^4 D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}}) \cdot \bar{j} \quad (120)$$

for open boundary condition and

$$A = e D_0^{\text{near}} \cdot J - [D_0^{\text{near}} - e^4 D_0^{\text{near}} \cdot W_D^{\text{el}(4)\text{near}} \cdot D_0^{\text{near}} + e^4 D_0^{\text{near}} \cdot \tilde{G}_0^{\text{near}} \cdot D_0^{\text{near}} \cdot \tilde{G}_0^{\text{near}} \cdot D_0^{\text{near}}] \cdot \bar{j} \quad (121)$$

for closed boundary condition.

Notice the difference between Eqs. (112) and (119)-(120), the equations of motion for the physical fields derived from the actions $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$ and $\Gamma[J, A]$ with open boundary conditions. We shall come back to this apparent paradox of having different equations for the same quantity in Section VII.

VI. MICROSCOPICAL CLASSICAL FIELD THEORY

It is important to distinguish two length scales when constructing classical field theories from quantum theory. The shorter one, $a = 2\pi\Lambda^{-1}$, is the UV cutoff of the underlying Quantum Field Theory. The other is the quantum-classical crossover length scale, ξ_{cr} , which is a rough order of magnitude estimate since the scale of the actual crossover depends on the environment. Classical field theories derived above for the expectation values of local operators provide information about the dynamics with a space-time resolution limited by a only. Structures seen at scales $a < \ell < \xi_{\text{cr}}$ characterize the microscopic quantum dynamics. Our classical field theory can be viewed as a time dependent generalization of the density functional theory [52–61] for relevant observables such as the electromagnetic current [62, 63] and the electromagnetic field. The current and electromagnetic field profiles, given by the equations of motion, help us to perform a partial resummation of the perturbation expansion. The study of the expectation values at scales $\ell \approx \xi_{\text{cr}}$ should give us some new insight into the quantum-classical transition because on the one hand, the description includes a large number of degrees of freedom needed to cope with classical objects and on the other hand, it is based on expectation values which are the relevant quantities at both sides of the transition. Notice that this is not the case when the usual, scattering amplitude based formalism of Quantum Field Theory is used because it is restricted to pure states and the decoherence can not even be formulated properly. Finally, we see macroscopic physics for $\xi_{\text{cr}} < \ell$.

Three aspects of microscopic classical theory will be mentioned in this Section. One is the decoherence and another, induced by it, the transfer of the time arrow by the environment. The third issue is the treatment of the polarization effects of the Dirac-see.

A. Decoherence of non-relativistic particles

Let us consider, as a simple but non-trivial problem, a system of N non-relativistic charges interacting with the photon field. The dynamics of charges is characterized by the action $S_c[\mathbf{x}]$ and the coupling to the electromagnetic field is realized via the current

$$j[\mathbf{x}]_{\mu x} = \sum_{n=1}^N \delta_{x^0, t} \delta_{\mathbf{x}, \mathbf{x}^{(n)}(t)} (1, \dot{\mathbf{x}}^{(n)}(t)) \quad (122)$$

where $\mathbf{x}^{(n)}(t)$ denotes the trajectory of the n -th charge. We consider the functional integral

$$Z = \int D[\hat{A}] D[\hat{\mathbf{x}}] e^{\frac{i}{2} \hat{A} \cdot \hat{D}_0^{-1} \cdot \hat{A} + i S[\hat{\mathbf{x}}] - i e \hat{j}[\mathbf{x}] \hat{\sigma} \cdot \hat{A}} \quad (123)$$

where $D[\hat{\mathbf{x}}]$ denotes the integral over the trajectories of all charges, $\hat{\mathbf{x}} = (\mathbf{x}^+, \mathbf{x}^-)$ stands for the particle trajectory, $S[\hat{\mathbf{x}}] = S[\mathbf{x}^+] - S^*[\mathbf{x}^-]$ and the current $\hat{j}[\mathbf{x}] = (j[\mathbf{x}^+], j[\mathbf{x}^-])$ couples to the photon field.

The photons can be eliminated in the usual way and the result is the action at a distance form of the electromagnetic interactions

$$Z = \int D[\hat{\mathbf{x}}] e^{i S_{\text{eff}}[\hat{\mathbf{x}}]}, \quad (124)$$

with

$$S_{\text{eff}}[\hat{\mathbf{x}}] = S[\hat{\mathbf{x}}] + W^{\text{phot}}[-e j[\mathbf{x}^+], e j[\mathbf{x}^-]] \quad (125)$$

where the last term on the right hand side is the influence functional [15]. The form (28) for the generating functional $W_{\text{OBC}}^{\text{phot}}[\hat{j}]$ gives

$$S_{\text{OBC}}[\hat{\mathbf{x}}] = S[\hat{\mathbf{x}}] - \frac{e^2}{2} j^\sigma[\mathbf{x}] \cdot D_0^{\text{ret}} \cdot \bar{j}^\sigma[\mathbf{x}] - \frac{e^2}{2} \bar{j}^\sigma[\mathbf{x}] \cdot D_0^{\text{adv}} \cdot j^\sigma[\mathbf{x}] - \frac{e^2}{2} j^\sigma[\mathbf{x}] (\kappa D_0^{\text{near}} + i \Im D_0) j^\sigma[\mathbf{x}] \quad (126)$$

for open boundary condition where the parameterization (14) of the current coupled by the matrix $\hat{\sigma}$ gives $j^\sigma = j^+ - j^-$ and $\bar{j}^\sigma = j^+(1 - \kappa)/2 + j^-(1 + \kappa)/2$. The relations (30), (40) allow us to rewrite the exponent of the integrand for open boundary condition in terms of the effective actions as

$$S_{\text{OBC}}[\hat{\mathbf{x}}] = S[\hat{\mathbf{x}}] + \Gamma_{\text{OBC}}^{\text{phot}}[A, A^{\text{adv}}] + i \Gamma_{\text{OBC}}^{\text{phot im}}[A^{\text{im}}] - e j^\sigma[\mathbf{x}] \cdot (A + i A^{\text{im}}) - e \bar{j}^\sigma[\mathbf{x}] \cdot A^{\text{adv}}, \quad (127)$$

where the fields A , A^{adv} and A^{im} are the solutions of the classical equations of motion (33), (42). The classical field theory based on the effective actions $\Gamma_{\text{OBC}}^{\text{phot}}[A, A^{\text{adv}}]$ and $\Im\Gamma_{\text{OBC}}^{\text{phot}}[A^{\text{im}}]$ allows us to obtain the Wilsonian effective action for the charges in terms of the expectation value of the electromagnetic field, A , and other non-physical expectation values, A^{adv} and A^{im} , induced by the trajectories $\hat{\mathbf{x}}$.

The imaginary part of the influence functional (126) is due to the continuous spectrum. It results from the construction of the reduced density matrix, the measurement of the unit operator, and represents a suppression

$$s = e^{\frac{e^2}{2}(j[\mathbf{x}^+] - j[\mathbf{x}^-]) \cdot \Im D_0 \cdot (j[\mathbf{x}^+] - j[\mathbf{x}^-])} \quad (128)$$

of the off-diagonal fluctuations in the density matrix $\rho(\mathbf{x}^+, \mathbf{x}^-)$. The imaginary part of the two current point function receive contributions from mass shell intermediate states, real photons, in this case. Non-accelerating charges do not emit photons; it is easy to check that $s = 1$ in this case.

Let us now consider a charge moving along a the trajectory $\mathbf{x}(t) = \mathbf{r} \cos \omega t$, the corresponding current being $j_{t,\mathbf{x}}^\mu = e\delta(\mathbf{x} - \mathbf{x}(t))(1, \dot{\mathbf{x}}(t))$. We shall calculate the imaginary part of the influence functional for $j_{t,\mathbf{x}}^\pm = j_{t,\mathbf{x} \pm \mathbf{R}/2}$, $r = |\mathbf{r}| \ll |\mathbf{R}| = R$ in Fourier space where

$$j_q^\mu = \left(eB(q^0, \omega, \mathbf{q}\mathbf{r}), \mathbf{r} \frac{ie\omega}{2} [B(q^0 - \omega, \omega, \mathbf{q} \cdot \mathbf{r}) - B(q^0 + \omega, \omega, \mathbf{q} \cdot \mathbf{r})] \right) \quad (129)$$

with

$$B(q^0, \omega, a) = \int_t e^{-iq^0 t + ia \cos \omega t} = \delta(q^0) + ia\delta(q^0 - \omega) + ia\delta(q^0 + \omega) + O(a^2), \quad (130)$$

and the suppression factor can be written as

$$s = e^{-2\pi \int_q \delta(q^2) |j_q|^2 \sin^2 \frac{\mathbf{q} \cdot \mathbf{R}}{2}}. \quad (131)$$

The diffraction integral in the exponent shows clearly the origin of the decoherence, the interference between the photons emitted by j^+ and j^- . The power series on the right hand side of Eq. (130) corresponds to the multipole expansion and we keep the dipole field only. Straightforward calculation gives in the limit $t_f - t_i \rightarrow \infty$

$$s_{\parallel} = e^{4e^2(t_f - t_i)r^2 \left[\left(\frac{\omega^2}{R} - \frac{2}{R^3} \right) \sin \omega R + \frac{2\omega}{R^2} \cos \omega R - \frac{\omega^3}{3} \right]} \quad (132)$$

when \mathbf{R} and \mathbf{r} are parallel. For $\mathbf{R} \cdot \mathbf{r} = 0$ one finds

$$s_{\perp} = e^{4e^2(t_f - t_i)r^2 \left[\frac{1}{R^3} \sin \omega r - \frac{\omega}{R^2} \cos \omega r - \frac{\omega^3}{3} \right]}. \quad (133)$$

The suppression factor interpolates smoothly between $s_{\parallel} = s_{\perp} = 1$ for $R = 0$ and

$$s_{\parallel} = s_{\perp} = e^{-\frac{4}{3}e^2(t_f - t_i)r^2\omega^3} \quad (134)$$

for $R \approx \infty$.

B. Radiation time arrow

The Lagrangian of QED is formally invariant with respect to the inversion of the direction of time, nevertheless our daily experience confirms that the currents generate retarded electromagnetic fields and the time reversal invariance is lost [64]. This symmetry breaking must come either from the environment, represented here as external sources which couple to the current and the electromagnetic field or from the way the dynamical problem is posed ie. the boundary conditions in time.

The CTP formalism is well suited for the investigation of this problem because of the duplication of the degrees of freedom. In the decomposition (7) of the field variable ϕ picks up the diagonal quantum fluctuations in the density matrix which appear in the expectation values of the canonical coordinates and ϕ^{adv} collects off-diagonal fluctuations, displayed by the expectation value of the canonical momentum variables.

But there is another way of looking at this separation: from the point of view of the time arrow. Let us return to the remark made in Section II that the time arrow is different for the coordinates ϕ^\pm appearing in the time ordered and anti-time ordered part in Eq. (3). Complications arise from the fact that matrix elements of time dependent

operators include both advanced and retarded effects. For instance, the Green function $\langle 0|T[\phi(x)\phi(y)]|0\rangle$ displays both advanced and retarded parts. The discussion of the linear response formulae in Section II makes it clear that any reflecting final boundary condition induces advanced fields. Can one separate the retarded and advanced effects? The answer is trivial and affirmative for non-interacting particles and the decomposition (7) gives ϕ and ϕ^{adv} as the combinations of the canonical coordinates which receive retarded or advanced effects of the external sources, respectively, for open boundary condition, according to Eqs. (35) and (36). The retarded nature of the physical field, ϕ , can be understood by noting that the in-field, observed at the initial time, is vanishing due to the choice $|\Psi_i\rangle = |0\rangle$. The separation of the advanced and retarded effects is trivial because it can be achieved by inspecting the initial (in) or final (out) field in the absence of vertices.

Interactions mix advanced and retarded effects because the non-linear pieces of the effective action in the fields are time reversal invariant. Consider for instance the equations of motion arising from the effective action $\Gamma[J, A]$ of Eq. (117). The time reversal invariance is broken by the environmental variables \bar{a} and \bar{j} which appear only in the terms linear in the fields, the part of the effective action which depends on the boundary conditions in time. The relation between the expectation values J and A , given by the boundary condition independent quadratic part of the effective action, involves the time reversal invariant near field propagator. The local expectation values appear to be insensitive to the symmetry breaking effects of the boundary condition. Our conjecture is that closed quantum systems maintain time reversal invariance for any boundary condition in time.

Such a claim makes the coupling to the classical environment responsible for the choice of the time arrow. But the environment characterized by the external sources \bar{a} and \bar{j} obey quantum laws, as well, suggesting that it is the classical limit for the environment which triggers the breakdown of the time reversal invariance. In fact, the decoherence which is supposed to be the hallmark of the quantum-classical crossover suppresses advanced fields without influencing retarded combinations, according to Eq. (126). Notice the double role the imaginary part of the influence functional plays in Eqs. (126)-(127). It sets not only the direction of time of the decoherence but determines the proportion of retarded and advanced parts in the charge propagator, too.

The equations of motion of the effective actions $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$ support the view that the retarded and advanced propagators appear in the equations of motion due to decoherence. On the first sight it is rather confusing that the quadratic part of the effective action in the fields involve the retarded and advanced Green functions when open boundary condition is used, in contrast to $\Gamma[J, A]$. How can different equations of motion occur at all for the same expectation values? The only difference between the equations of motion for J and A , arising from the effective actions $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$ and $\Gamma[J, A]$ is that different quantities are kept fixed during the variations. In fact, let us construct a variation of A by varying the external sources a and j for fixed \bar{a} and \bar{j} in such a manner that J remains unchanged. Such a variation gives rise to the equation of motion for A , Eq. (120), of the effective action $\Gamma[J, A]$. In the case of the effective action $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$ all four sources are varied by keeping all field but A fixed. The lesson is that the time arrow appeared in the equations of motion relating measured expectation values because the variations respected the condition $J^{\text{adv}} = A^{\text{adv}} = 0$, the vanishing of the off-diagonal fluctuations. This is exactly what decoherence is for.

Can the decoherence solve the arrow of time problem? Not, it makes one step only in reducing it to the environment. When the photons are eliminated in Section VIA then their boundary condition in time fixes the sign of $\Im D_0$ and consequently the arrow of time for the charges. What is remarkable is that this transfer of symmetry breaking becomes enhanced as we enter the macroscopic domain by decreasing the cutoff. This suggests that a better understanding of the arrow of time problem can be obtained by the application of the renormalization group method to this problem.

C. Vacuum polarization

In classical electrodynamics the polarizability of a medium is taken into account in the well known, simple manner supposing the higher multipole moments are negligible. It is interesting to see what kind of approximation corresponds to this procedure when the polarizability of the Dirac-see is considered.

The effective action for the electromagnetic field,

$$\Gamma[A] = \frac{1}{2}A \cdot (D_0^{\text{near}-1} - e^2 \tilde{G}_0^{\text{near}-1} + e^2 \tilde{G}_0^{\text{near}-1} \cdot W_D^{\text{el}(4)\text{near}} \cdot \tilde{G}_0^{\text{near}-1}) \cdot A - eA \cdot J^{\text{ext}} + O(A^3) + O(e^4), \quad (135)$$

obtained by eliminating the current J by its equation of motion and reintroducing the elementary charge e in the minimal coupling in Eq. (117). We ignore the gauge fixing term because current conservation decouples the gauge sector and write $\Gamma[A] = \tilde{\Gamma}[F]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ due to gauge invariance. A conserved current is defined as

$$j = -\frac{1}{e} \frac{\delta \Gamma[A]}{\delta A} \Big|_{A=0} = \frac{1}{e} \left(\partial^\nu \frac{\delta \tilde{\Gamma}[F]}{\delta F^{\mu\nu}} - \partial^\nu \frac{\delta \tilde{\Gamma}[F]}{\delta F^{\nu\mu}} \right) \Big|_{F=0}. \quad (136)$$

The minimal coupling is constructed by the introduction of the tensor $H_{\mu\nu} = \partial_\mu j_\nu - \partial_\nu j_\mu$ for the current and writing the $O(A)$ part of the effective action as

$$-eA \cdot j = \frac{e}{2} F_{\mu\nu} \cdot \square^{-1} \cdot H^{\mu\nu} \quad (137)$$

where the conservation of the currents was used on the right hand side. The field strength tensor which includes the vacuum polarization is defined as

$$G = -2 \left(\frac{\delta \tilde{\Gamma}[F]}{\delta F} - \frac{\delta \tilde{\Gamma}[F]}{\delta F} \Big|_{F=0} \right). \quad (138)$$

To find the linearized equation of motion for the photon field it is sufficient to have the accuracy $O(J)$ and $O(A^2)$ for the effective action,

$$\Gamma[A] = -\frac{1}{4} F \cdot G - eA \cdot j + O(A^3). \quad (139)$$

To find a more detailed expression for the field strength tensor G return to the effective action $\Gamma[A]$ and assume that its $O(A^2)$ part is translational invariant. This latter property allows us to write

$$\frac{1}{2} A \cdot T \cdot \Gamma^{(2)} \cdot T \cdot A = -\frac{1}{8} F_{\mu\nu} \cdot \frac{1}{\square} [\Gamma^{(2)}, F]_+^{\mu\nu} - \frac{1}{8} F_{\mu\rho} \cdot L^\rho{}_\nu \cdot \frac{1}{\square} [\Gamma^{(2)}, F]_-^{\mu\nu} + \frac{1}{8} F_{\rho\mu} \cdot L^\rho{}_\nu \cdot \frac{1}{\square} [\Gamma^{(2)}, F]_-^{\mu\nu} \quad (140)$$

where

$$([\Gamma, F]_\pm)^{\mu\nu} = \int_y [\Gamma_{x-y}^{\mu\rho} F_{\rho}{}^\nu{}_y \pm F_y^{\mu\rho} \Gamma_{\rho}{}^\nu{}_{x-y}] \quad (141)$$

the integration over the space-time coordinate y being shown explicitly. We write $\Gamma^{(2)} = -D_0^{\text{near}-1} + \Sigma$, cf. Eq. (135), Σ being the photon self energy, and find

$$G = F + \frac{1}{2} [\Sigma, \square^{-1} F]_+ - \frac{1}{2} L \cdot [\Sigma, \square^{-1} F]_- + \frac{1}{2} (L \cdot [\Sigma, \square^{-1} F]_-)^{\text{tr}} \quad (142)$$

where the superscript tr denotes the exchange of the two space-time indices in the last term and renders G an antisymmetric tensor.

It is instructive to obtain the field strength tensor in three-dimensional notations when $A_\mu = (\phi, -\mathbf{A})$, $\mathbf{E} = -\partial_0 \mathbf{A} - \nabla \phi$, $\mathbf{B} = \nabla \times \mathbf{A}$ and the parameterization

$$\Sigma_{\mu\nu} = \begin{pmatrix} \Sigma_0 & \boldsymbol{\Sigma} \\ \boldsymbol{\Sigma} & \boldsymbol{\Pi} \end{pmatrix} \quad (143)$$

yield

$$G^{\mu\nu} = F + \frac{1}{2} \square^{-1} \begin{pmatrix} 0 & (\boldsymbol{\Pi} - \Sigma_0) \mathbf{E} - \boldsymbol{\Sigma} \times \mathbf{B} \\ (\Sigma_0 - \boldsymbol{\Pi}) \mathbf{E} - \boldsymbol{\Sigma} \times \mathbf{B} & \boldsymbol{\Sigma} \otimes \mathbf{E} - \mathbf{E} \otimes \boldsymbol{\Sigma} - [\boldsymbol{\Pi}, \Phi[\mathbf{B}]]_+ \end{pmatrix} + \frac{1}{2} \square^{-2} \begin{pmatrix} 0 & \partial(2\partial_0 \boldsymbol{\Sigma} \cdot \mathbf{E} + \boldsymbol{\partial} \cdot \mathbf{v}) - \partial_0(\partial_0 \mathbf{u} + \boldsymbol{\partial} w) \\ -\partial(2\partial_0 \boldsymbol{\Sigma} \cdot \mathbf{E} + \boldsymbol{\partial} \cdot \mathbf{v}) + \partial_0(\partial_0 \mathbf{u} + \boldsymbol{\partial} w) & \partial_0 \partial^k u^j - \partial_0 \partial^j u^k + \partial^\ell \partial^k w^{\ell j} - \partial^\ell \partial^j w^{\ell k} \end{pmatrix} \quad (144)$$

with

$$\begin{aligned} \mathbf{u} &= -(\boldsymbol{\Pi} + \Sigma_0) \mathbf{E} - \boldsymbol{\Sigma} \times \mathbf{B} \\ \mathbf{v} &= -(\boldsymbol{\Pi} + \Sigma_0) \mathbf{E} + \boldsymbol{\Sigma} \times \mathbf{B} \\ w &= \boldsymbol{\Sigma} \otimes \mathbf{E} + \mathbf{E} \otimes \boldsymbol{\Sigma} - [\boldsymbol{\Pi}, \Phi[\mathbf{B}]]_- \end{aligned} \quad (145)$$

$\Phi[\mathbf{B}]_{jk} = \epsilon_{jkl} B_\ell$ being the spacelike part of the bare field strength tensor F .

The motion of the charges in the Dirac sea mixes the electric and the magnetic fields in the vacuum polarization if translational invariance is assumed only. The rather involved form of the polarization shown in Eq. (144) results from the Schwinger-Dyson resummation of the one-loop photon self energy parameterized as (143). It is interesting to

note that similar resummation has already been used in classical electrodynamics in arriving at the Clausius-Mossotti formule [65].

The polarization simplifies considerably when rotational invariance, $\Pi_{jk} = \Pi\delta_{jk}$, $\Sigma = 0$ imposed. Expression (144) results in

$$\mathbf{D} = \mathbf{E} + \square^{-1} \left(\Sigma_0 - \frac{1}{2} \Sigma_{\text{nr}} \cdot [1 + \square^{-1}(\partial_0^2 - \partial \otimes \partial)] \right) \mathbf{E} \quad (146)$$

and

$$\mathbf{H} = \mathbf{B} + \square^{-1} \left[(\Sigma_0 - \Sigma_{\text{nr}}) \mathbf{B} + \frac{1}{2} \Sigma_{\text{nr}} \square^{-1} \partial_0 \partial \times \mathbf{E} \right], \quad (147)$$

respectively, where \mathbf{D} and \mathbf{H} are the electric and the magnetic fields contained by the field strength tensor G , and $\Sigma_{\text{nr}} = \Pi + \Sigma_0$.

Further simplification is achieved by imposing Lorentz invariance, $\Sigma_{\text{nr}} = 0$, because the electric and magnetic sectors no longer mix. The dielectric tensor ϵ and magnetic permeability tensor μ defined by the relations $\mathbf{D} = \epsilon \cdot \mathbf{E}$ and $\mathbf{B} = \mu \cdot \mathbf{H}$, respectively, are

$$\epsilon = \mu^{-1} = \mathbb{1} + \square^{-1} \Sigma_0, \quad (148)$$

reproducing well known one-loop result [66]. Notice that the operator \square^{-1} on the right hand side is the result of gauge invariance, the relation between the gauge field and the field strength tensor.

The n -th multipole moments of the vacuum polarization are constructed by projecting the $\ell = n$ angular momentum components out of the current n -point Green function. The dipole contribution corresponds to the retaining of the $O(a^2)$ part of the generating functional $W[a, j]$. The order of the retained dependence on j gives the accuracy in terms of the powers of e , according to Eq. (C3). The effective action formalism where the Legendre transformation performs the inversion of the quadratic part, the resummation of the Schwinger-Dyson equation, yields the relation between the applied and the polarized fields. The total field strength (144) is based on the dipole contribution to the polarization taken into account in the same order in e as the photon self energy (143). Higher multipole moments introduce higher order terms in the fields in the effective action and lead to non-linear polarization effects. Note that the n -th order multipole contributions do not appear before the order n of the expansion in e .

We have traced the effects of vacuum polarization in the real part of the influence functional for the photon field which give the quantum corrections to the classical Maxwell equations. The imaginary part of this functional eg. $\Im \epsilon$ in a relativistically invariant theory plays important role, as well, by introducing decoherence and setting the time arrow for the electromagnetic field.

VII. QUANTUM RENORMALIZATION GROUP

The problem of the quantum-classical transition can be studied by adjusting the resolution of the space-time averages which is usually realized within the framework of the renormalization group method. This method was developed in its full generalities first when the Kadanoff-Wilson blocking was applied to the description of critical phenomena in classical Statistical Physics [67]. The application of blocking in Quantum Field Theory followed immediately and agreement with the already known multiplicative renormalization group scheme was established.

However, there is an essential difference between the successive elimination of degrees of freedom in a classical and a quantum system which was ignored. In classical statistical physics the form of the partition function can be maintained during the blocking but the vacuum-to-vacuum transition amplitude of a quantum system is changed in a fundamental manner when particle modes are eliminated. In fact, the transition amplitude between two states corresponds to a process occurring for pure states while in Quantum Mechanics the elimination of a degree of freedom generates mixed states to be described by a density matrix. This goes beyond the usual formalism based on "in-out" type vacuum-to-vacuum transition amplitudes.

Let us follow the blocking procedure on the path integral (8) where some high energy particle modes, $p > \Lambda$, are integrated over. As long as the the initial and final states are pure the density matrices factorize in the variables ϕ^\pm and the successive integration over particle modes of the fields ϕ^\pm preserves this factorization. Notice that such a blocking corresponds to the elimination of the particle modes which possess given initial and final states coded by the density matrices. Let us suppose, for example, that the occupation numbers for the high energy modes to be eliminated are fixed in the initial and final states. Then the resulting path integral displays a dynamics for the low energy particles which depends on these occupation numbers.

This is not what is meant by elimination of a degrees of freedom in Quantum Mechanics. The measurement of the identity operator, the calculation of the trace of the density matrix in the factor space of the degrees of freedom to be eliminated, destroys all information about the initial and final states. The information about the final state is destroyed by the trace operation, the dependence on the initial state disappears due to the unitarity of the time evolution.

It is this point, the realization of such a true elimination of degrees of freedom without retaining information about the initial and final states, where the CTP formalism becomes essential. We erase the final state information for the high energy modes to be eliminated by installing the identity operator as density matrix in that sector of the Fock space. The identity operator, the functional Dirac-delta for the given modes of $\phi^+ - \phi^-$ couples the variables along the two time axis and this coupling is handed over for the low energy modes. In other words, once we perform a true trace operation on the high energy sector of the Fock space the remaining low energy states become mixed. This mixing requires the use of the CTP formalism for the rest of the blocking procedure.

The difference between the blocking in the vacuum-to-vacuum amplitude and in the CTP path integral is that in the former case the trace operation is performed in the zero particle number sector of the high energy modes only. This represents a small error when the the particle modes to be eliminated are at very high energy compared with the observational scale and perturbation expansion applies. But in non-asymptotically free models where the high energy dynamics is non-perturbative or in effective theories with not too high cutoff Λ the weight of the particles at the cutoff scale is non-negligible and the difference between these two blocking becomes more important. It is the quantum renormalization group procedure, based on the CTP formalism, what should be followed in these cases.

There is no difficulty in extending the functional renormalization group scheme [68, 69] to the CTP formalism [39, 40] but the doubling of the degrees of freedom renders the resulting renormalized trajectory formal. A more natural formalism, based on the measurable expectation values and the degrees of freedom controlling decoherence, is outlined in Appendix E. But instead of embarking a detailed study of this evolution equation we shall be satisfied by a few qualitative remarks about the emergence of classical physics. The renormalized trajectory of realistic models passes at several fixed points and may experience different scaling laws separated by crossover regimes [70]. One of the crossovers belongs to the quantum-classical transition. The difference between the trajectories generated by the blocking in the traditional, scattering amplitude based formalism and the CTP path integral should be less important at the UV side of the quantum-classical crossover. The decoherence which requires the density matrix is supposed to be the hallmark of the quantum-classical crossover. Therefore one expects that this transition and possible others further down towards the infrared direction are missed completely or reproduced with substantial errors by the "in-out" formalism which can not handle mixed states. Note that we need a bosonized form of the theory where the excitations are handled by bosonic fields in order to embark the problem of the quantum-classical crossover because the expectation values of fermionic operators are trivially vanishing.

The decoherence suppresses the fluctuations of the field ϕ^{adv} . But it is just the first moment, the expectation value of this kind of fields which is suppressed by construction in deriving the variational equations of motion of the effective action $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$. This suggests that these equations which already assume decoherence change less at the quantum-classical crossover than the variational equations of the effective action $\Gamma[J, A]$ which do involve non-vanishing advanced fields.

It is worth mentioning that the $O(A^2)$ part of the action $\Gamma[J, A]$, given by Eq. (117), has already been suggested many years ago [48–50]. By analogy of the steps leading from Eq. (100) to Eq. (105) we find for vanishing external sources the well known action at a distance

$$\kappa\Gamma[J] = -\frac{1}{2}J \cdot \tilde{G}^{\text{near}-1} \cdot J = \Gamma^{\text{mech}}[J] + \frac{e^2}{2}J \cdot D_0^{\text{near}} \cdot J, \quad (149)$$

with time reversal invariant current-current interaction term. The earlier proposals were made on the basis of classical electrodynamics where the charges follow world lines and the self-interaction was excluded in order to avoid UV divergences. In Eq. (149) self-interaction is included. Such theories have no radiation field, accelerating charges induce near field only. The retarded and advanced fields appear with the same weight and therefore the action-reaction balance is obviously satisfied. The well established retarded radiation field can be recovered by taking into account the reaction of all charges to the electromagnetic field in a completely absorbing Universe [71]. As mentioned above, the boundary conditions in time influence the linear part of the action only. As a result, the $O(A^2)$ part is the same for open ended time evolution and for fixed boundary condition when the photons end up at the vacuum state in a completely absorbing Universe. In other words, the classical argument of Ref. [71] applies to open boundary condition imposed on the quantum level, as well. The time arrow and the Abraham-Lorentz force [46, 72–74] are generated by the presence of other than the accelerating charge in the system. But this argument is valid in the microscopic domain only. As soon as the cutoff is lowered and our action is given in the classical domain, decoherence is implied automatically and the time arrow is generated by the retarded Liénard-Wiechert potential of the action $\Gamma[J, J^{\text{adv}}, A, A^{\text{adv}}]$ and the Abraham-Lorentz force is generated by the accelerating charge itself.

The renormalization group approach reveals a similarity between spontaneous symmetry breaking and the arrow of time problem. Spontaneous symmetry breaking appears in the renormalized trajectories as a crossover to an infrared scaling regime which contains relevant operators of lower symmetry. The boundary condition in time imposed on photons is passed over the charges when photons starts to behave classically at the quantum-classical crossover. At the final count, the arrow of time problem appears as a spontaneous breakdown of the time inversion symmetry. As such, it indicates that an extreme sensitivity develops in the system among the infrared modes to choose a time arrow from "outside". The identity of thermodynamical arrows formed in different domains and the importance of the cosmological master arrow can be explained by such a mechanism.

We close this Section by pointing out an unexpected difficulty arising at the quantum-classical crossover. When reaching the crossover by decreasing the cutoff the advanced coordinates become suppressed. The semiclassical limit corresponds to the dominance of the path integral by the configurations $\phi^{\text{adv}} = 0$. But there is still another variable, ϕ , to integrate over. Notice that there is a cancellation between the actions $S[\phi^+]$ and $S^*[\phi^-]$ in the exponent on the right hand side of Eq. (8) when $\phi^{\text{adv}} = 0$ causing the integrand of the path integral to become flat. This appears in the renormalization group scheme as a strong coupling regime for the configurations ϕ . Thus, decoherence induces strong coupling behavior and our approximations obtained within the framework of the loop-expansion break down. Even the degrees of freedom may change.

This scenario is in strong contrast to the semiclassical limit discussed usually in the framework of the path integral formalism. The limit $\hbar \rightarrow 0$ of the single time axis non-relativistic path integral

$$\int D[\phi] e^{\frac{i}{\hbar} S[\phi]} \quad (150)$$

is treated by the stationary phase approximation arguing that the integral is dominated by the paths close to the classical trajectory. This description applies to a closed quantum system which is not realistic in the macroscopic limit. When a density matrix is introduced to describe the separation of the system from its environment, the time axis is doubled and the CTP path integral

$$\int D[\phi^+] D[\phi^-] e^{\frac{i}{\hbar} S[\phi^+] - \frac{i}{\hbar} S^*[\phi^-] + \frac{i}{\hbar} S_{\text{infl}}[\phi^+, \phi^-]} \quad (151)$$

is used where $S_{\text{infl}}[\phi^+, \phi^-]$ denotes the influence functional of the environment. This expression is not dominated anymore by the classical trajectory. Instead there is an approximate cancellation between the two time axis when decoherence sets in, $S_{\text{infl}} \approx M^2(\phi^+ - \phi^-)^2/2$ with large M^2 and the physical field, ϕ , is integrated over with approximately flat integrand. The classical limit is supposed to be recovered on the level of the equations of motion for the expectation values of observables only. The approximation to the transition amplitude (150) in the semiclassical limit is based on the rigidity of the dominant trajectory, the excitations being suppressed by $1/\hbar$. In contrast of this simple picture the semiclassical limit is supposed to be reached when the energy level density becomes high. This view takes into account the environment and is supported by the strong coupling, soft saddle point scenario of the functional integral (151) of the CTP formalism.

VIII. SUMMARY

The dynamics of the expectation value of local operators in QED, the electric current and the electromagnetic field in particular, was studied in this work. The expectation values were given in the framework of the CTP formalism covering either fixed initial and final states or open, unconstrained time evolution.

It was shown that the equations of motion for the expectation values of local operators can be obtained as variational equations of a suitably defined effective action mixing the diagonal and off-diagonal quantum fluctuations in field diagonal basis. The effective action plays a double role. It is not only a classical action for a classical field theory for the expectation values of local operators but it appears also as a Wilsonian effective action for degrees of freedom coupled to the system by our observables.

The effective actions for free photons and electrons are calculated first. While the former is a trivial exercise the latter proves to be a highly involved problem due to the multi-particle aspects of the Dirac-see. The dynamics of localized charged states depends on the states which contribute to the current expectation value in a manner reminiscent of small and large polarons of Solid State Physics. On the one hand, negative energy single-particle states contributing to the polarization generated by weak external field overlap strongly. As a result of the dynamics of two-particle states generated by the current from the vacuum, Pauli-blocking generates a long-range correlation which produces separation-independent forces preventing the separation of localized charges by weak external electromagnetic field in the non-interacting Dirac-see. On the other hand, the charges arising from localized states within the mass gap

decouple from the filled up negative energy states and from each other. For such charges the standard relativistic action for localized particles was recovered. The actual value of the mass parameter depends on the details of the bound state and it gives rise a renormalization condition on the classical field theory level.

The effective action for the interacting electron-photon system was constructed up to quadratic parts in the fields on the two-loop order. The elimination of either the current or the electromagnetic field produces the one-loop action at a distance theory. The quadratic part of the one-loop effective action for the current and the electromagnetic field calculated in the leading order of the gradient expansion agrees with the action of classical electrodynamics. Quantum corrections to the action come from three directions. Vacuum polarizations due to electron-photon interactions generate form-factors and new, higher order contributions in the fields which can be treated perturbatively having e^2 as small parameter. Other e -independent quantum corrections appear due to the indistinguishability of electrons in the non-interacting Dirac-see. The small parameter to organize an expansion in the exchange effects is the absolute magnitude of the ratio of the electron Compton wavelength square and the invariant length square of the separation in space-time. Finally, the third class of contributions corresponds to the boundary conditions in time which are build in at a microscopic level.

The Dirac-see is the natural polarizable medium of QED and its effects can be taken into account in the same manner as in classical electrodynamics, by the introduction of the displacement fields and dispersive electric permeability, when the action of the classical field theory is truncated in the quadratic approximation.

Though the effective actions mentioned in this work govern expectation values of a single measurement only they give some qualitative informations as far as multiple non-demolishing measurements are concerned. The correlation function of two successive non-demolishing measurements of the current is the inverse of the $O(J^2)$ kernel of the effective action. In general, the multiple current measurement results are weakly correlated by vacuum-polarization effects when they are carried out in space-time locations separated by an invariant lengths whose absolute magnitude exceeds the Compton wavelength.

The CTP formalism is particularly well suited to trace the way the quantum boundary conditions in time generate the time arrow in classical physics. A conjecture was put forward, namely that closed quantum systems maintain time reversal invariance for any boundary conditions in time. When the system becomes open by coupling it to external sources the response is given in terms of retarded Green functions and a time arrow is generated for unconstrained, open ended time evolution. It was found that advanced effects mix in at some order only if the system is not allowed to follow an open ended time evolution and is projected on some states at the final time. Advanced effects represent an influence of the future on the present and are possible in Quantum Mechanics due to the superposition principle which allows us to specify both initial and final states. The symmetrical treatment of two time axis with opposite sense of time in the CTP highlights the fact that such effects are as natural as the influence of the initial condition on the course of the motion. Similar phenomenon exists in classical physics, as well. By specifying the initial and final coordinates of a classical system the change of the final coordinate modifies the whole time evolution. But this is a triviality because the state of the system became uniquely defined at the final point only, the equations of motion being of second order. Notice that the influence of the boundary conditions in time does not decrease when the distance in time between an observation and the boundary conditions is increased, in other words there is no clusterisation in time due the unitarity of the time evolution.

The CTP formalism duplicates each degree of freedom, $\phi \rightarrow \phi^\pm$, for the treatment of the density matrix $\rho(\phi^+, \phi^-)$ and the two copies are subject to opposite time arrow. The sum $\phi^+ + \phi^-$ (difference $\phi^+ - \phi^-$) remains invariant (changes sign) under the inversion of the time and follows retarded (advanced) dynamics. The observed averages are given by a certain linear combination of ϕ^+ and ϕ^- and classical field theories constructed for the measured fields only can not identify the sense of time. As a result, the dynamics of such theories is time reversal invariant by mixing retarded and advanced effects with equal weight. The retarded and advanced effects can be separated by retaining both ϕ^+ and ϕ^- in the classical dynamics. The resulting dynamics reflects decoherence on the average because $\langle \Psi | \phi^+ - \phi^- | \Psi \rangle = 0$. The classical field theory constructed for both ϕ^+ and ϕ^- predicts indeed retarded dynamics for the measured combination. Decoherence, the suppression of $\phi^+ - \phi^-$, generates macroscopical time arrow by suppressing degrees of freedom subject to advanced dynamics.

The effective action formalism offers classical dynamics both for the microscopic and macroscopic regimes because the space-time resolution of the expectation values is limited by the UV cutoff Λ . For high enough Λ we see the microscopic quantum structure of local operators and the effective action is an extension of the density functional theory. As the cutoff is lowered one recovers a hydrodynamics description, classical field theory for the current, and macroscopic physics. The issue of quantum-classical crossover can be addressed in this generalization of the renormalization group scheme for the density matrix. The decoherence is specially clear in the CTP formalism where it appears as a simple one-loop effect. It is conjectured that the quantum-classical crossover which is characterized by decoherence is actually a strong coupling phenomenon for the physical fields.

The qualitative similarity between suppression of advanced combination of the degrees of freedom and decoherence suggests that the equations of motions of classical field theories which trace both degrees of freedom of the CTP

formalism changes less at the quantum-classical crossover than those of the classical theories containing the measurable fields only. Thus the equations of motion for the measurable field interpolate between the time symmetrical form of the classical theory for the physical fields only and the retarded equations of the classical theory for all fields as Λ is decreased and cuts through the quantum-classical crossover. This reasoning shows the peculiarity of the Abraham-Lorentz force of radiation damping. This force originates from the consistent motion of all charges, interacting by the near-field Liénard-Wiechert potentials when the space-time resolution of the expectation values is microscopic. But once the space-time resolution is chosen to be macroscopic the Abraham-Lorentz force arises from the accelerating "blocked" charge alone. Such a view of radiation damping applies to dissipative forces in general and explains the need of auxiliary variables when such effects are incorporated into canonical dynamics. Another remarkable fact is that the quadratic part of the action for the physical fields is independent of the boundary conditions in time. Consider two Universes corresponding to the same initial conditions but one is allowed to have an open, unconstrained time evolution while the state of the other is projected on the vacuum at a final time. The latter represents a completely absorbing Universe and its linearized equations of motions are identical with that of the Universe with open ended time evolution.

The application of Quantum Field Theory to establish the dynamics of the expectation values offers a comprehensive framework for measurement theory. The basic difficulties of the measurement theory is to identify well established and understood quantum effects in the interaction of a small and a large system. The difficulty comes from the presence of a dimensionless number, n/N , the ratio of the number of degrees of freedom in the two systems taking extremely small values and producing unexpected effects in a manner analogous to Statistical Physics. The n/N -dependence of the expectation values can be traced by the study of the cutoff-dependence because the small system can be imagined as residing in the elementary volume element of the underlying field theory model. The cutoff-dependence can systematically be obtained by eliminating the degrees of freedom, by the application of the renormalization group strategy based on a suitable chosen blocking procedure. The quantum-classical crossover shows some formal similarity with spontaneous symmetry breaking, both representing new, less symmetrical scaling laws in the infrared scaling regime. The radiational time arrow is passed between subsystems by this mechanism. In the case of QED the time reversal invariance for the charges is broken by the boundary conditions in time imposed for photons at the quantum-classical crossover. This mechanism is realized by the imaginary part of the photon propagator which is non-vanishing for continuous spectrum, ie. the infinite time limit only. Another, conjectured feature of the quantum-classical crossover, the strongly coupled dynamics, might be part of the problem which prevents us from having a better insight into the quantum-classical transition.

Acknowledgments

This work could not have been carried out without the stimulating discussions with Janos Hajdu whose critical reading of the manuscript was helpful, too. Furthermore I thank Kornel Sailer for the careful check of some formulae and Andor Frenkel for pointing out the relevance of refs. [13, 14].

APPENDIX A: TWO-POINT FUNCTION FOR HERMITIAN SCALAR FIELD

In this Appendix the generating functional is constructed for a hermitian scalar subject to boundary conditions $\rho_i = |0\rangle\langle 0|$ and $\rho_f = \mathbb{1}$. We start by collecting some formulae for the propagators of any hermitian, spinless, local operator ϕ . The retarded, advanced near and far field Green functions are introduced in the next step, followed by a brief recall of the spectral representation. Finally, the propagator of a free, neutral scalar particle is discussed and the massless results are given in closed form.

1. Propagator as a block-matrix

The connected Green functions are obtained by means of the generating functional

$$e^{iW[\hat{j}]} = \text{Tr} \bar{T} [e^{i \int_{t_i}^{t_f} dt' j^-(t') \phi_-(t')}] T [e^{i \int_{t_i}^{t_f} dt' j^+(t') \phi_+(t')}] |0\rangle\langle 0| \quad (\text{A1})$$

and the propagator is defined as a 2×2 block-matrix

$$\Delta_{xy}^{\sigma\sigma'} = - \frac{\delta^2 W[\hat{j}]}{\delta j_x^\sigma \delta j_y^{\sigma'}} \Big|_{\hat{j}=0}. \quad (\text{A2})$$

When both legs of the propagator are on the positive-sense time axis one finds the usual causal propagator

$$\begin{aligned} i\Delta_{xy}^{++} &= \sum_n \langle n | T[\phi_x \phi_y] | 0 \rangle \langle 0 | n \rangle \\ &= \langle 0 | T[\phi_x \phi_y] | 0 \rangle. \end{aligned} \quad (\text{A3})$$

The propagator along the negative-sense time axis is

$$\begin{aligned} i\Delta_{xy}^{--} &= \sum_n \langle n | 0 \rangle \langle 0 | \bar{T}[\phi_x \phi_y] | n \rangle \\ &= \langle 0 | T[\phi_y \phi_x] | 0 \rangle^* \\ &= (i\Delta_{yx}^{++})^*. \end{aligned} \quad (\text{A4})$$

Finally, the mixed propagator is given by

$$i\Delta_{xy}^{+-} = \sum_n \langle 0 | \phi_y | n \rangle \langle n | \phi_x | 0 \rangle. \quad (\text{A5})$$

The choice $\rho_f = \mathbb{1}$ allows us to decrease t_f until it reaches $\max(x^0, y^0)$ and placing both field operators into the same expectation value resulting

$$i\Delta_{xy}^{+-} = \langle 0 | \phi_y \phi_x | 0 \rangle. \quad (\text{A6})$$

Note that the identity $T[\phi_x \phi_y] + \bar{T}[\phi_x \phi_y] = \phi_x \phi_y + \phi_y \phi_x$ leads to the relation

$$\Delta^{++} + \Delta^{--} = \Delta^{+-} + \Delta^{-+}. \quad (\text{A7})$$

2. Retarded and advanced propagators

Introducing the notation $\Delta = \Delta^{++}$ for the causal propagator

$$i\Delta_{x,x'} = \Theta(t - t') \langle 0 | \phi_x \phi_{x'} | 0 \rangle + \Theta(t' - t) \langle 0 | \phi_{x'} \phi_x | 0 \rangle, \quad (\text{A8})$$

its real and imaginary parts are

$$\Re \Delta_{x,x'} = -\frac{1}{2} \Theta(t - t') i \langle 0 | [\phi_x, \phi_{x'}] | 0 \rangle - \frac{1}{2} \Theta(t' - t) i \langle 0 | [\phi_{x'}, \phi_x] | 0 \rangle = -\frac{1}{2} \epsilon(t - t') i \langle 0 | [\phi_x, \phi_{x'}] | 0 \rangle \quad (\text{A9})$$

with $\epsilon(t) = \text{sign}(t)$ and

$$\Im \Delta_{x,x'} = -\frac{1}{2} \Theta(t - t') \langle 0 | \{\phi_x, \phi_{x'}\} | 0 \rangle - \frac{1}{2} \Theta(t' - t) \langle 0 | \{\phi_{x'}, \phi_x\} | 0 \rangle = -\frac{1}{2} \langle 0 | \{\phi_x, \phi_{x'}\} | 0 \rangle, \quad (\text{A10})$$

respectively. The retarded and advanced propagators are defined as usual,

$$i\Delta_{xx'}^{\bar{a}} = \pm \Theta(\pm(t - t')) \langle 0 | [\phi_x, \phi_{x'}] | 0 \rangle. \quad (\text{A11})$$

It is easy to find the actual expressions in terms of $\hat{\Delta}$ by noting

$$\begin{aligned} i\Delta^r &= \Theta(t - t') \langle 0 | \phi_x \phi_{x'} | 0 \rangle + \Theta(t' - t) \langle 0 | \phi_{x'} \phi_x | 0 \rangle \\ &= i\Delta_{xx'} - \Theta(t' - t) \langle 0 | \phi_{x'} \phi_x | 0 \rangle - \Theta(t - t') \langle 0 | \phi_{x'} \phi_x | 0 \rangle \\ &= i\Delta_{xx'} - i\Delta_{xx'}^{+-} \\ &= i\Delta_{xx'}^* + i\Delta_{xx'}^{-+}, \end{aligned} \quad (\text{A12})$$

where the relation $i\Delta^{--} = (i\Delta^{++})^*$ and Eq. (A7) were used in obtaining the last equation. The near and far field propagators, $\Delta^n = \Re \Delta$ and $\Delta^f = 2\Re \Delta^{-+}$ satisfy the equation

$$\Delta^r = \Delta^n + \frac{1}{2} \Delta^f. \quad (\text{A13})$$

Similar reasoning yields the equation

$$\Delta^a = \Delta^n - \frac{1}{2}\Delta^f \quad (\text{A14})$$

for the advanced propagator. The complete propagator can be parameterized by means of three real functions, for instance

$$\hat{\Delta} = \begin{pmatrix} \Delta^n + i\Im\Delta & -\frac{1}{2}\Delta^f + i\Im\Delta \\ \frac{1}{2}\Delta^f + i\Im\Delta & -\Delta^n + i\Im\Delta \end{pmatrix} \quad (\text{A15})$$

3. Spectral representation

A particularly useful parameterization of the propagator is provided by the spectral representation. By following the standard procedure introduce the eigenvectors of the energy-momenta, $P^a|n\rangle = p_n^a|n\rangle$, to induce the space-time dependence $\phi_x = e^{iPx}\phi_0 e^{-iPx}$ and write

$$\begin{aligned} i\langle 0|\phi_x\phi_y|0\rangle &= \sum_n i\langle 0|\phi_x|n\rangle\langle n|\phi_{x'}|0\rangle \\ &= \sum_n i|\langle 0|\phi_0|n\rangle|^2 e^{-ip_n(x-x')} \\ &= i \int_p e^{-ip(x-y)} \Theta(p^0) \rho(p^2), \end{aligned} \quad (\text{A16})$$

where

$$\Theta(p^0)\rho(p^2) = \frac{1}{2\pi} \sum_n (2\pi)^4 \delta(p_n - p) |\langle 0|\phi_0|n\rangle|^2, \quad (\text{A17})$$

is the spectral function. One can recast the last line of Eqs. (A16) in the form reminiscent of free propagators,

$$\begin{aligned} i\langle 0|\phi_x\phi_y|0\rangle &= i \int_p \int_0^\infty d\mu^2 e^{-ip(x-x')} \Theta(p^0) \rho(\mu^2) 2\pi \delta(p^2 - \mu^2) \\ &= i \int_{\vec{k}} \int_0^\infty d\mu^2 e^{-i\omega_{\vec{k}}(\mu^2)(t-t') + i\vec{k}(\mathbf{x}-\mathbf{x}')} \rho(\mu^2), \end{aligned} \quad (\text{A18})$$

where $\omega_{\vec{k}}(\mu^2) = \sqrt{\mu^2 + k^2}$ and

$$\int_{\vec{k}} = \int \frac{d\vec{k}}{(2\pi)^3 2\omega_{\mu^2, \vec{k}}}. \quad (\text{A19})$$

The propagator Δ is obtained by means of the Fourier representation of the Heavyside-function,

$$\Delta_{xx'} = - \int_0^\infty d\mu^2 \rho(\mu^2) \int_{\vec{k}, \omega} \left(\frac{e^{-i(\omega_{\vec{k}}(\mu^2) + \omega)(t-t') + i\vec{k}(\mathbf{x}-\mathbf{x}')}}{\omega + i\epsilon} + \frac{e^{i(\omega_{\vec{k}}(\mu^2) + \omega)(t-t') - i\vec{k}(\mathbf{x}-\mathbf{x}')}}{\omega + i\epsilon} \right). \quad (\text{A20})$$

and performing the change of variable $\omega \rightarrow -\omega$ in the second integral

$$\begin{aligned} \Delta_{xx'} &= - \int_0^\infty d\mu^2 \rho(\mu^2) \int_{\vec{k}, \omega} \left(\frac{e^{-i(\omega_{\vec{k}}(\mu^2) + \omega)(t-t') + i\vec{k}(\mathbf{x}-\mathbf{x}')}}{\omega + i\epsilon} + \frac{e^{i(\omega_{\vec{k}}(\mu^2) - \omega)(t-t') + i\vec{k}(\mathbf{x}-\mathbf{x}')}}{-\omega + i\epsilon} \right) \\ &= \int_0^\infty d\mu^2 \rho(\mu^2) \Delta_{0 \ xx'}(\mu^2) \end{aligned} \quad (\text{A21})$$

where

$$\Delta_{0 \ xx'}(\mu^2) = - \int_{\vec{k}} \frac{e^{-i\vec{k}(\mathbf{x}-\mathbf{x}')}}{k^2 - \mu^2 + i\epsilon}. \quad (\text{A22})$$

The steps above repeated for the block-propagator yield

$$\hat{\Delta} = \int_0^\infty d\mu^2 \rho(\mu^2) \hat{\Delta}_0(\mu^2). \quad (\text{A23})$$

4. Free neutral scalar particle

The path integral representation of non-interacting generating functional is

$$\begin{aligned} e^{iW_0[\hat{j}]} &= \text{Tr} \bar{T} [e^{i \int_{t_i}^{t_f} dt' [H(t') + \int_{\mathbf{x}} \hat{j}_{t', \mathbf{x}}^- \phi_{t', \mathbf{x}}^-]}] T [e^{-i \int_{t_i}^{t_f} dt' [H(t') - \int_{\mathbf{x}} \hat{j}_{t', \mathbf{x}}^+ \phi_{t', \mathbf{x}}^+]}] |0\rangle \langle 0| \\ &= \int D[\hat{\phi}] e^{\frac{i}{2} \hat{\phi} \cdot \hat{\Delta}_0^{-1} \cdot \hat{\phi} + i \hat{j} \cdot \hat{\phi}}. \end{aligned} \quad (\text{A24})$$

where $\hat{\Delta}_0 = \hat{\Delta}_0(m^2)$. The path integral can be carried out with the result

$$W_0[\hat{j}] = -\frac{1}{2} \hat{j} \cdot \hat{\Delta}_0 \cdot \hat{j}. \quad (\text{A25})$$

The detailed form of the propagators can be obtained by means of the Fourier integral

$$\phi_x = \int_{\mathbf{k}} [e^{-i\omega_{\mathbf{k}}t + i\mathbf{k}\mathbf{x}} a_{\mathbf{k}} + e^{i\omega_{\mathbf{k}}t - i\mathbf{k}\mathbf{x}} a_{\mathbf{k}}^\dagger] \quad (\text{A26})$$

where $\omega_{\mathbf{k}} = \omega_{\mathbf{k}}(m^2)$. The non-vanishing canonical commutation relation are $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 2\omega_{\mathbf{k}} \delta_{\mathbf{k}, \mathbf{k}'}$. The repetition of the steps (A20)-(A21) leads to the causal propagator (A22).

5. Real space expressions for $m = 0$

First we calculate the expectation value

$$\begin{aligned} \langle 0 | \phi_x \phi_{x'} | 0 \rangle &= \int_{\mathbf{k}} e^{-ik(x-x')} \\ &= \frac{1}{8\pi^2} \int_0^\infty dk k \int_{-1}^1 d \cos \theta e^{-ikt + ikr \cos \theta}. \end{aligned} \quad (\text{A27})$$

The contributions to the k -integration are negligible at the upper limit when working in a cutoff theory which can be incorporated by calculating the integral for $t \rightarrow t - i\epsilon$ yielding

$$\langle 0 | \phi_x \phi_{x'} | 0 \rangle = -\frac{1}{4\pi^2} \frac{1}{(x-x')^2 - i\epsilon(x-x')^0 \epsilon} \quad (\text{A28})$$

and

$$D_{0 \ x x'} = \frac{1}{4\pi} \delta((x-x')^2) - \frac{i}{4\pi^2} P \frac{1}{(x-x')^2}, \quad (\text{A29})$$

$D_0 = \Delta_0(0)$ being the free massless propagator. The retarded and advanced propagators are

$$D_{0 \ x x'}^{\bar{a}} = \frac{1}{2\pi} \Theta(\pm(x^0 - x'^0)) \delta((x-x')^2). \quad (\text{A30})$$

APPENDIX B: TWO-POINT VERTEX FUNCTION FOR NON-INTERACTING DIRAC-SEE

In this Appendix we present the calculation of the kernel of the quadratic effective action for the current with $\beta = 0$ for the non-interacting Dirac-see.

The Fourier representation of the quadratic form (71)

$$\Gamma_{(t_1, \mathbf{x}_1), (t_2, \mathbf{x}_2)}^{(2)\text{el}} = -\frac{15\pi m^2}{(2\pi)^4} [\Theta(t_1 - t_2) + \Theta(t_2 - t_1)] \int_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)} \int_{k_0} \frac{e^{ik_0(t_1 - t_2)}}{(k_0^2 - \mathbf{k}^2 + i\epsilon)^2} \quad (\text{B1})$$

gives after straightforward integration over the frequency and the polar angles

$$\epsilon(t) \Gamma^{(2)\text{el}}(t, r) = \frac{15m^2}{8r} [itf_1(t+r) - itf_1(t-r) + f_2(t+r) - f_2(t-r)] \quad (\text{B2})$$

with $t = t_1 - t_2$, $r = |\mathbf{x}_1 - \mathbf{x}_2|$ and

$$f_n(t) = \frac{1}{2\pi} \int_k \frac{e^{-ikt}}{k^n}. \quad (\text{B3})$$

The relation $\partial_t^n f_n(t) = (-i)^n \delta(t)$ yields $f_1(t) = c_1 - i\Theta(t)$ and $f_2(t) = -t\Theta(t) + tc_2 + c_3$, c_k being integration constants to be determined. The vertex function

$$\Gamma^{(2)\text{el}}(t, r) = \frac{\epsilon(t)15m^2}{8} [2c_2 - \Theta(t-r) - \Theta(t+r)] \quad (\text{B4})$$

obtained by replacing the solution for $f_1(t)$ and $f_2(t)$ into Eq. (B2) should be symmetric in t therefore $c = 1/2$ and Eq. (72) follows.

APPENDIX C: LOOP EXPANSION FOR $W[\hat{a}, \hat{j}]$

In this Appendix some details of the calculation of the generating functional for connected Green functions of the current and the photon field in QED are presented.

1. Setting up the loop expansion

We start by Eq. (87) for the generating functional written after the photon field is integrated out,

$$e^{iW[\hat{a}, \hat{j}]} = e^{iW^{\text{el}}[\hat{a} + ie\hat{\sigma} \frac{\delta}{\delta \hat{j}}] + iS_{CT}[-ie\frac{\delta}{\delta \hat{j}}]} e^{-\frac{i}{2}\hat{j} \cdot \hat{D}_0 \cdot \hat{j}}. \quad (\text{C1})$$

The identity

$$\frac{\delta}{\delta f} e^{F[f]} \mathbb{1} = e^{F[f]} \left(\frac{\delta}{\delta f} + \frac{\delta F[f]}{\delta f} \right) \mathbb{1} \quad (\text{C2})$$

yields

$$\begin{aligned} e^{iW[\hat{a}, \hat{j}]} &= e^{-\frac{i}{2}\hat{j} \cdot \hat{D}_0 \cdot \hat{j}} e^{iW^{\text{el}}[\hat{a} + ie\hat{\sigma} \frac{\delta}{\delta \hat{j}} + e\hat{\sigma} \hat{D}_0 \cdot \hat{j}] + iS_{CT}[-ie\frac{\delta}{\delta \hat{j}} - e\hat{D}_0 \cdot \hat{j}]} \mathbb{1} \\ &= e^{-\frac{i}{2}\hat{j} \cdot \hat{D}_0 \cdot \hat{j}} \sum_{m=0}^{\infty} \frac{i^m}{m!} \left[\sum_{n=0}^{\infty} \frac{i^n}{n!} \left(e \frac{\delta}{\delta \hat{j}} \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} \right)^n W^{\text{el}}[\hat{a} + \hat{a}'] \right]^m \mathbb{1} \end{aligned} \quad (\text{C3})$$

where

$$\hat{a}' = e\hat{\sigma} \hat{D}_0 \cdot \hat{j} \quad (\text{C4})$$

and $\delta'/\delta \hat{a}'$ acts only on the first factor of W^{el} . For simplicity the counterterms are suppressed. The \hat{j} -dependence comes from \hat{a}' therefore we have

$$e^{iW[\hat{a}, \hat{j}]} = e^{-\frac{i}{2}\hat{j} \cdot \hat{D}_0 \cdot \hat{j}} \sum_{m=0}^{\infty} \frac{i^m}{m!} \left[\sum_n \frac{1}{n!} \left(\frac{\delta}{\delta \hat{a}'} \cdot ie^2 \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} \right)^n W^{e'}[\hat{a} + \hat{a}'] \right]^m \mathbb{1}. \quad (\text{C5})$$

2. Two-loop order quadratic generating functional

We shall calculate $W[\hat{a}, \hat{j}]$ on the two-loop order up to quadratic terms in the external sources. To this end we need

$$W = W^{\text{el}}[\hat{a} + \hat{a}'] + \frac{\delta}{\delta \hat{a}'} \cdot ie^2 \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} W^{\text{el}}[\hat{a} + \hat{a}'] + \frac{1}{2} \left(\frac{\delta}{\delta \hat{a}'} \cdot ie^2 \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} \right)^2 W^{\text{el}}[\hat{a} + \hat{a}'] \quad (\text{C6})$$

and its square in Eq. (C5) up to two-loop terms. We start with the expansion in \hat{a}' ,

$$W^{\text{el}}[\hat{a} + \hat{a}'] \approx W^{\text{el}}[\hat{a}] + W_a^{\text{el}(1)}[\hat{a}] \hat{a}'_a + \frac{1}{2} W_{ab}^{\text{el}(2)}[\hat{a}] \hat{a}'_a \hat{a}'_b + \frac{1}{3!} W_{abc}^{\text{el}(3)}[\hat{a}] \hat{a}'_a \hat{a}'_b \hat{a}'_c + \frac{1}{4!} W_{abcd}^{\text{el}(4)}[\hat{a}] \hat{a}'_a \hat{a}'_b \hat{a}'_c \hat{a}'_d \quad (\text{C7})$$

by retaining the $O(\hat{a}^4)$ pieces only. Two functional derivations give

$$\begin{aligned}\frac{\delta}{\delta \hat{a}'_a} W^{\text{el}}[\hat{a} + \hat{a}'] &\approx W_a^{\text{el}(1)}[\hat{a}] + W_{ab}^{\text{el}(2)}[\hat{a}] \hat{a}'_b + \frac{1}{2} W_{abc}^{\text{el}(3)}[\hat{a}] \hat{a}'_b \hat{a}'_c + \frac{1}{3!} W_{abcd}^{\text{el}(4)}[\hat{a}] \hat{a}'_b \hat{a}'_c \hat{a}'_d \\ \frac{\delta^2}{\delta \hat{a}'_a \delta \hat{a}'_b} W^{\text{el}}[\hat{a} + \hat{a}'] &\approx W_{ab}^{\text{el}(2)}[\hat{a}] + W_{abc}^{\text{el}(3)}[\hat{a}] \hat{a}'_c + \frac{1}{2} W_{abcd}^{\text{el}(4)}[\hat{a}] \hat{a}'_c \hat{a}'_d,\end{aligned}\quad (\text{C8})$$

where the coefficients are

$$\begin{aligned}W^{\text{el}(1)}[\hat{a}] &\approx \hat{j}[\hat{a}] = W^{\text{el}(2)} \hat{a}, \\ W_{ab}^{\text{el}(2)}[\hat{a}] &\approx W_{ab}^{\text{el}(2)} + \frac{1}{2} W_{abcd}^{\text{el}(4)} \hat{a}_c \hat{a}_d, \\ W_{abc}^{\text{el}(3)}[\hat{a}] &\approx W_{abcd}^{\text{el}(4)} \hat{a}_d, \\ W_{abcd}^{\text{el}(4)}[\hat{a}] &\approx W_{abcd}^{\text{el}(4)} + \frac{1}{2} W_{abcdef}^{\text{el}(6)} \hat{a}_e \hat{a}_f\end{aligned}\quad (\text{C9})$$

in the given order. The photon exchange appears in the combinations

$$\begin{aligned}W_D^{\text{el}(2)}[\hat{a}] &= W_{ab}^{\text{el}(2)}[\hat{a}] (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{ab} \\ W_D^{\text{el}(3)}[\hat{a}]_a &= W_{abc}^{\text{el}(3)}[\hat{a}] (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{bc} \\ W_D^{\text{el}(4)}[\hat{a}]_{ab} &= W_{abcd}^{\text{el}(4)}[\hat{a}] (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{cd}.\end{aligned}\quad (\text{C10})$$

The expressions

$$\begin{aligned}\frac{\delta}{\delta \hat{a}'} \cdot i \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} W^{\text{el}}[\hat{a} + \hat{a}'] &\approx W_D^{\text{el}(2)}[\hat{a}] + W_D^{\text{el}(3)} \cdot \hat{a}' + \frac{1}{2} \hat{a}' \cdot W_D^{\text{el}(4)}[\hat{a}] \cdot \hat{a}' \\ &\quad + \left(W_a^{\text{el}(1)}[\hat{a}] + W_{ab}^{\text{el}(2)}[\hat{a}] \hat{a}'_b + \frac{1}{2} W_{abc}^{\text{el}(3)}[\hat{a}] \hat{a}'_b \hat{a}'_c + \frac{1}{e 3!} W_{abcd}^{\text{el}(4)}[\hat{a}] \hat{a}'_b \hat{a}'_c \hat{a}'_d \right) (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{ae} \frac{\delta}{\delta \hat{a}'_e}, \\ \left(\frac{\delta}{\delta \hat{a}} \cdot \hat{\sigma} i 2 \hat{D}_0 \hat{\sigma} \cdot \frac{\delta'}{\delta \hat{a}'} \right)^2 W^{\text{el}}[\hat{a} + \hat{a}'] &\approx \left(W_{Da}^{\text{el}(3)}[\hat{a}] + W_{Dab}^{\text{el}(4)}[\hat{a}] \hat{a}'_b \right) (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{ae} \frac{\delta}{\delta \hat{a}'_e}.\end{aligned}\quad (\text{C11})$$

give

$$\begin{aligned}W &= W^{\text{el}}[\hat{a}] + W^{\text{el}(1)}[\hat{a}] \cdot \hat{a}' + \frac{1}{2} \hat{a}' \cdot W_{ab}^{\text{el}(2)}[\hat{a}] \cdot \hat{a}' + \frac{1}{3!} W_{abc}^{\text{el}(3)}[\hat{a}] \hat{a}'_a \hat{a}'_b \hat{a}'_c + \frac{1}{4!} W_{abcd}^{\text{el}(4)}[\hat{a}] \hat{a}'_a \hat{a}'_b \hat{a}'_c \hat{a}'_d \\ &\quad + e^2 W_D^{\text{el}(2)}[\hat{a}] + e^2 W_D^{\text{el}(3)} \cdot \hat{a}' + \frac{e^2}{2} \hat{a}' \cdot W_D^{\text{el}(4)}[\hat{a}] \cdot \hat{a}'\end{aligned}\quad (\text{C12})$$

in the given truncation. Its square

$$W^2 \approx \left(W^{\text{el}}[\hat{a}] + W_a^{\text{el}(1)}[\hat{a}] \hat{a}'_a + \frac{1}{2} W_{ab}^{\text{el}(2)}[\hat{a}] \hat{a}'_a \hat{a}'_b \right)^2 + e^2 (\hat{a} + \hat{a}') \cdot W^{(2)}[\hat{a}] \cdot \hat{\sigma} i \hat{D}_0 \hat{\sigma} \cdot W^{(2)}[\hat{a}] \cdot (\hat{a} + \hat{a}') \quad (\text{C13})$$

is obtained by taking into account unitarity, $W^{\text{el}}[0] = 0$. We have now everything needed to re-exponentiate the necessary terms in Eq. (C5),

$$\begin{aligned}1 + iW - \frac{1}{2} W^2 &\approx \exp i \left[W^{\text{el}}[\hat{a}] + W^{\text{el}(1)}[\hat{a}] \cdot \hat{a}' + \frac{1}{2} \hat{a} \cdot W^{\text{el}(2)}[\hat{a}] \cdot \hat{a}' + e^2 W_D^{\text{el}(2)}[\hat{a}] + e^2 W_D^{\text{el}(3)} \cdot \hat{a}' + \frac{e^2}{2} \hat{a}' \cdot W_D^{\text{el}(4)}[\hat{a}] \cdot \hat{a}' \right. \\ &\quad \left. - \frac{e^2}{2} (\hat{a} + \hat{a}') \cdot W^{\text{el}(2)}[\hat{a}] \cdot \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot W^{\text{el}(2)}[\hat{a}] \cdot (\hat{a} + \hat{a}') \right]\end{aligned}\quad (\text{C14})$$

yielding

$$\begin{aligned}W[\hat{a}, \hat{j}] &= -\frac{1}{2} \hat{j} \cdot \hat{D}_0 \cdot \hat{j} - \frac{1}{2} (\hat{a} + \hat{a}') \cdot \hat{G}_0 \cdot (\hat{a} + \hat{a}') + \frac{e^2}{2} W_{abcd}^{\text{el}(4)} \hat{a}_c \hat{a}_d (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{ab} \\ &\quad + e^2 W_{abcd}^{\text{el}(4)} \hat{a}'_a \hat{a}_d (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{bc} + \frac{e^2}{2} W_{abcd}^{\text{el}(4)} (\hat{\sigma} i \hat{D}_0 \hat{\sigma})_{cd} \hat{a}'_a \hat{a}'_b - \frac{e^2}{2} (\hat{a} + \hat{a}')_a \hat{G}_{0ab} (\hat{\sigma} \hat{D}_0 \hat{\sigma})_{be} \hat{G}_{0ef} (\hat{a} + \hat{a}')_f.\end{aligned}\quad (\text{C15})$$

The substitution of \hat{a}' given by Eq. (C4) results in

$$W[\hat{a}, \hat{j}] = \frac{1}{2} \hat{j} \cdot [-\hat{D}_0 - e^2 \hat{D}_0 \hat{\sigma} \cdot (\hat{G}_0 - e^2 W_D^{(4)}) \cdot \hat{\sigma} \hat{D}_0 - e^4 \hat{D}_0 \hat{\sigma} \cdot \hat{G}_0 \cdot \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \hat{G}_0 \cdot \hat{\sigma} \hat{D}_0] \cdot \hat{j} \quad (C16)$$

$$+ \frac{1}{2} \hat{a} (-\hat{G}_0 + e^2 W_D^{(4)} - e^2 \hat{G}_0 \cdot \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \hat{G}_0) \hat{a} + e \hat{j} [-\hat{D}_0 \hat{\sigma} \cdot (\hat{G}_0 - e^2 W_D^{(4)}) - e^2 \hat{D}_0 \hat{\sigma} \cdot \hat{G}_0 \cdot \hat{\sigma} \hat{D}_0 \hat{\sigma} \cdot \hat{G}_0] \hat{a}$$

leading to expressions (88)-(92) for the generating functional. The parameterization (14) and (67) give

$$\Re W[a, \bar{a}, j, \bar{j}] = \frac{1}{2} (a, j, \bar{a}, \bar{j}) \cdot \begin{pmatrix} \frac{\kappa}{2} (K + K^{\text{tr}}) & K \\ K^{\text{tr}} & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ j \\ \bar{a} \\ \bar{j} \end{pmatrix} \quad (C17)$$

with

$$K_{\text{OBC}} = - \begin{pmatrix} \tilde{G}^{\text{ret}} & e \tilde{G}^{\text{ret}} D_0^{\text{ret}} \\ e D_0^{\text{ret}} \tilde{G}^{\text{ret}} & D^{\text{ret}} \end{pmatrix} \quad (C18)$$

for OBC shown in a detailed form in Eq. (95). For FBC the sources belonging to different time axis decouple giving Eq. (C17) with

$$K_{\text{FBC}} = - \begin{pmatrix} \tilde{G}^{\text{near}} & e(\tilde{G}^{\text{near}} D_0^{\text{near}} + \Im \tilde{G} \Im D_0) \\ e(D_0^{\text{near}} \tilde{G}^{\text{near}} + \Im D_0 \Im \tilde{G}) & D_0 \end{pmatrix}. \quad (C19)$$

For the sake of completeness we record the imaginary part,

$$\Im W_{\text{OBC}}[a, j] = -\frac{1}{2} (a, j) \cdot \begin{pmatrix} \Im \tilde{G} & e(\Im \tilde{G} D_0^{\text{adv}} + \tilde{G}^{\text{ret}} \Im D_0) \\ e(D_0^{\text{ret}} \Im \tilde{G} + \Im D_0 \tilde{G}^{\text{adv}}) & \Im D \end{pmatrix} \cdot \begin{pmatrix} a \\ j \end{pmatrix}, \quad (C20)$$

and

$$\Im W_{\text{FBC}}[a, \bar{a}, j, \bar{j}] = -\frac{1}{2} (a, \bar{a}, j, \bar{j}) \cdot \begin{pmatrix} \Im \tilde{G} \begin{pmatrix} \frac{1+\kappa^2}{2} & \kappa \\ \kappa & 2 \end{pmatrix} & e(\Im \tilde{G} D_0^{\text{near}} + \tilde{G}^{\text{near}} \Im D_0) \begin{pmatrix} \frac{1+\kappa^2}{2} & \kappa \\ \kappa & 2 \end{pmatrix} \\ e(D_0^{\text{near}} \Im \tilde{G} + \Im D_0 \tilde{G}^{\text{near}}) \begin{pmatrix} \frac{1+\kappa^2}{2} & \kappa \\ \kappa & 2 \end{pmatrix} & \Im \hat{D} \begin{pmatrix} \frac{1+\kappa^2}{2} & \kappa \\ \kappa & 2 \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} a \\ \bar{a} \\ j \\ \bar{j} \end{pmatrix}. \quad (C21)$$

APPENDIX D: LEGENDRE TRANSFORMATION

The calculation of the quadratic effective action is sketched in this Appendix. There is a useful relation between the second derivatives of a function and its Legendre transform. Let us denote all sources and fields in the problem by j and ϕ . Then the effective action is defined by $\Gamma[\phi] = W[j] - j \cdot \phi$ with $\phi = \delta W[j]/\delta j$. One can easily prove the identity

$$\frac{\delta^2 W[j]}{\delta j \delta j} \cdot \frac{\delta^2 \Gamma[\phi]}{\delta \phi \delta \phi} = -\mathbb{1} \quad (D1)$$

which renders the calculation of the Legendre transform of a quadratic expression to an inversion problem.

We start with a few useful relations for the inversion of block-matrices to be used later and rules for obtaining the retarded and advanced components from a product of propagators. This is followed by the calculation of the effective action for four fields with open and fixed boundary conditions.

1. Inversion of block-propagators

We shall meet two kinds of inversion during the Legendre transformation of different functionals. One of them is the inversion of a 2×2 block propagator

$$K = - \begin{pmatrix} \tilde{G} & e S_1 \\ e S_2 & D \end{pmatrix}. \quad (D2)$$

The inverse

$$\begin{pmatrix} -\tilde{G} & -eS_1 \\ -eS_2 & -D \end{pmatrix}^{-1} = - \begin{pmatrix} \tilde{G}^{\text{int}-1} & -e\tilde{G}^{-1}S_1D^{\text{int}-1} \\ -eD^{\text{int}-1}S_2\tilde{G}^{-1} & D^{\text{int}-1} \end{pmatrix} \quad (\text{D3})$$

where

$$\tilde{G}^{\text{int}} = \tilde{G} - e^2 S_1 D^{-1} S_2 \quad (\text{D4})$$

and

$$D^{\text{int}} = D - e^2 S_2 \tilde{G}^{-1} S_1 \quad (\text{D5})$$

is obtained by solving the system of linear equations

$$\begin{pmatrix} a \\ b \end{pmatrix} = K \begin{pmatrix} x \\ y \end{pmatrix} \quad (\text{D6})$$

by elimination.

Another inversion problem one encounters involves the block-matrix

$$\begin{pmatrix} \frac{\kappa}{2}(K + K^{\text{tr}}) & K \\ K^{\text{tr}} & 0 \end{pmatrix}. \quad (\text{D7})$$

The same strategy as in the previous case ie. the solution of the corresponding set of linear equations gives

$$\begin{pmatrix} \frac{\kappa}{2}(K + K^{\text{tr}}) & K \\ K^{\text{tr}} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & K^{\text{tr}-1} \\ K^{-1} & -\frac{\kappa}{2}(K^{\text{tr}-1} + K^{-1}) \end{pmatrix}. \quad (\text{D8})$$

2. Light-cone as a null-space

The real part of massless particle propagators are non-vanishing on the light-cone only. This property simplifies enormously the expressions of the perturbation expansion frequently involving a chain of products of massless propagators $\Pi = \hat{D}_1 \hat{\sigma} \cdot \hat{D}_2 \hat{\sigma} \cdots \hat{D}_n$ and its retarded and advanced parts.

We start with the demonstration of the useful relation

$$D^{\text{far}} \cdot D^{\text{near}-1} = 0 \quad (\text{D9})$$

valid for massless propagators. The straightforward proof of Eq. (D9) is by inserting the propagator

$$D_{xx'}^{\text{ret}} = - \int_k \frac{e^{-ik(x-x')}}{(k^0 \pm i\epsilon)^2 - \mathbf{k}^2} F(k^2), \quad (\text{D10})$$

with $F(0) = 1$ resulting

$$(D^{\text{far}} \cdot D^{\text{near}-1})_{xx'} = -i\epsilon \int_k P e^{-ik(x-x')} \frac{\epsilon(k^0)}{k^2} \quad (\text{D11})$$

which is indeed vanishing for $\epsilon \rightarrow 0$.

To understand better the regularization effects of ϵ we note that on the one hand $D_0 = (D^{\text{ret}} + D^{\text{adv}})/2$ satisfies the inhomogeneous equation of motion

$$\frac{1}{F(-\square)} \cdot \square \cdot D_0 = \mathbb{1}. \quad (\text{D12})$$

Thus, $G^{\text{near}} \rightarrow \infty$ and $G^{\text{near}-1} \rightarrow 0$ on the light cone for $\epsilon \rightarrow 0$. On the other hand, D^{far} is non-vanishing on the light cone only. Therefore, Eq. (D9) follows. A more explicit argument is based on the fact that D^{far} is a homogeneous Green function satisfying the equation

$$\frac{1}{F(-\square)} \cdot \square \cdot D^{\text{far}} = 0 \quad (\text{D13})$$

which is just Eq. (D9) due to Eq. (D12).

The lesson is the identity $D^{\text{near}-1} \cdot D^{\text{ret}} = D^{\text{near}-1} \cdot D^{\text{adv}}$, ie. the equation $D^{\text{ret}} = D^{\text{adv}}$ holds when the propagators appear in a product beside of a factor $D^{\text{near}-1}$. This result yields

$$D^{\text{near}-1} \cdot D^{\text{ret}} = D^{\text{near}-1} \cdot D^{\text{adv}} = \mathbb{1}. \quad (\text{D14})$$

Let us finally consider the product Π where each propagator is of the form

$$D_j = \begin{pmatrix} 0 & D_j^{\text{adv}} \\ D_j^{\text{ret}} & i\Im D_j \end{pmatrix} \quad (\text{D15})$$

in the basis $(j^+ - j^-, j^+ + j^-)$. It is easy to verify that Π is of this form, as well, and

$$\begin{aligned} \hat{D}_1 \hat{\sigma} \cdots \hat{D}_n &= \begin{pmatrix} 0 & D_1^{\text{adv}} \cdot D_2^{\text{adv}} \cdots D_n^{\text{adv}} \\ D_1^{\text{ret}} \cdot D_2^{\text{ret}} \cdots D_n^{\text{ret}} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & \Im D_1 \cdot D_2^{\text{adv}} \cdots D_n^{\text{adv}} + D_2^{\text{ret}} \cdot \Im D_2 \cdots D_n^{\text{adv}} \cdots + D_1^{\text{ret}} \cdot D_2^{\text{ret}} \cdots \Im D \end{pmatrix} \end{aligned} \quad (\text{D16})$$

giving

$$\begin{aligned} (\hat{D}_1 \hat{\sigma} \cdots \hat{D}_n)^r &= D_1^{\text{ret}} \cdots D_n^{\text{ret}}, \\ (\hat{D}_1 \hat{\sigma} \cdots \hat{D}_n)^a &= D_1^{\text{adv}} \cdots D_n^{\text{adv}}. \end{aligned} \quad (\text{D17})$$

The property (D14) can be used to show

$$\frac{1}{2}(D_1^{\text{ret}} \cdots D_n^{\text{ret}} + D_1^{\text{adv}} \cdots D_n^{\text{adv}}) = D_1^{\text{near}} \cdots D_n^{\text{near}} \quad (\text{D18})$$

giving

$$(\hat{D}_1 \hat{\sigma} \cdots \hat{D}_n)^{\text{near}} = D_1^{\text{near}} \cdots D_n^{\text{near}}. \quad (\text{D19})$$

The fact that the product $\hat{D}_1 \hat{\sigma} \cdots \hat{D}_n$ preserves the form (D15) assures that $W_D^{\text{el}(4)}$ which appears in the self energy is of this form, too.

Eqs. (D17) and (D19) can be used to show that once the pair creation is excluded ($\Im \tilde{G} = 0$) the generating functional $W_{\text{FBC}}[\hat{a}, \hat{j}]$ is obtained from $W_{\text{OBC}}[\hat{a}, \hat{j}]$ by the simple rule found for non-interacting systems, namely the replacement of all retarded or advanced propagator by their near field version.

3. Effective actions for four fields

To obtain the effective action for the fields J , J^{adv} , A and A^{adv} we need the inverse of the block-matrices of Eqs. (95) and (96). We start with OBC when the two-loop expression electron self energy in Eq. (91) and the first equation in (D17) give

$$\tilde{G}^{\text{ret}} = G_0^{\text{ret}} + e^2 \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} - e^2 W_D^{\text{el}(4)\text{ret}}. \quad (\text{D20})$$

In a similar manner the photon self energy of Eq. (92) gives

$$D^{\text{ret}} = \hat{D}_0^{\text{ret}} + e^2 D_0^{\text{ret}} \cdot (\tilde{G}_0^{\text{ret}} - e^2 W_D^{\text{el}(4)\text{ret}}) \cdot D_0^{\text{ret}} + e^4 D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}} \cdot \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}} \quad (\text{D21})$$

the inverse being

$$D^{\text{ret}-1} = D_0^{\text{ret}-1} - e^2 (\tilde{G}_0^{\text{ret}} - W_D^{\text{el}(4)\text{ret}}). \quad (\text{D22})$$

These expressions substituted in Eqs. (D4) and (D5) give $D^{\text{int}} = \hat{D}_0^{\text{ret}}$ and $\tilde{G}^{\text{int}} = G_0^{\text{ret}} - e^2 W_D^{\text{el}(4)\text{ret}}$. The inverse (D3) gives the expression (111) when choosing $S_1 = \tilde{G}_0^{\text{ret}} \cdot D_0^{\text{ret}} = S_2^{\text{tr}}$. Finally, the effective action (110) follows from Eqs. (D1) and (D8) and its equations of motion give

$$\begin{aligned} \tilde{G}_0^a a &= -(1 + e^2 W_D^{\text{el}(4)a}) \cdot \tilde{G}_0^{a-1} \cdot J^{\text{adv}} + e \tilde{G}_0^a \cdot A^{\text{adv}}, \\ D_0^{\text{adv}} j &= e D_0^{\text{adv}} \cdot J^{\text{adv}} - A^{\text{adv}}, \\ \tilde{G}_0^{\text{ret}} \cdot \left(\bar{a} + \frac{\kappa}{2} a \right) &= -(1 + e^2 W_D^{\text{el}(4)\text{ret}} \cdot \tilde{G}_0^{\text{ret}-1}) \cdot \left(J - \frac{\kappa}{2} J^{\text{adv}} \right) + e \tilde{G}_0^{\text{ret}} \cdot \left(A - \frac{\kappa}{2} A^{\text{adv}} \right), \\ D_0^{\text{ret}} \cdot \left(\bar{j} + \frac{\kappa}{2} j \right) &= e D_0^{\text{ret}} \cdot \left(J - \frac{\kappa}{2} J^{\text{adv}} \right) - A + \frac{\kappa}{2} A^{\text{adv}}. \end{aligned} \quad (\text{D23})$$

According to the remark made at the end of Appendix D 2 the effective action $\Gamma_{\text{FBC}}[J, J^{\text{adv}}, A, A^{\text{adv}}]$ is obtained from $\Gamma_{\text{FBC}}[J, J^{\text{adv}}, A, A^{\text{adv}}]$ by the replacement of all retarded or advanced propagators by their near field version as long as pair creation is neglected.

4. Effective actions for two fields

The effective action which involves physical fields results from the Legendre transformation in the variables a and j only. The generating functional $\Re W[a, \bar{a}, j, \bar{j}]$ based on the two-point function (95) or (96) is

$$\Re W[a, \bar{a}, j, \bar{j}] = -\frac{\kappa}{2}(a, j) \cdot \begin{pmatrix} \tilde{G}^{\text{near}} & e\tilde{G}^{\text{near}} \cdot D_0^{\text{near}} \\ e\tilde{G}^{\text{near}} \cdot D_0^{\text{near}} & D_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ j \end{pmatrix} + (a, j) \cdot \begin{pmatrix} W_a \\ W_j \end{pmatrix} \quad (\text{D24})$$

where the only boundary condition dependent pieces are

$$W_{\text{OBC}a} = -\tilde{G}^{\text{ret}} \cdot (\bar{a} + eD_0^{\text{ret}} \cdot \bar{j}) \quad W_{\text{OBC}j} = -D_0^{\text{ret}} \cdot \bar{j} - eD_0^{\text{ret}} \cdot \tilde{G}^{\text{ret}} \cdot \bar{a}, \quad (\text{D25})$$

or

$$W_{\text{FBC}a} = -\tilde{G}^{\text{near}} \cdot (\bar{a} + eD_0^{\text{near}} \cdot \bar{j}) \quad W_{\text{FBC}j} = -D_0 \cdot \bar{j} - eD_0^{\text{near}} \cdot \tilde{G}^{\text{near}} \cdot \bar{a}. \quad (\text{D26})$$

The application of the inverse (D3) produces the effective action

$$\Gamma_{\text{FBC}}[J, A] = -\frac{1}{2\kappa}(J, A) \cdot K^{-1} \cdot \begin{pmatrix} J \\ A \end{pmatrix} + \frac{1}{\kappa}(J, A) \cdot K^{-1} \cdot \begin{pmatrix} W_a \\ W_j \end{pmatrix} \quad (\text{D27})$$

with

$$K^{-1} = -\frac{1}{\kappa} \begin{pmatrix} \tilde{G}_0^{\text{near}-1} + \tilde{G}_0^{\text{near}-1} \cdot W_D^{(4)n} \cdot \tilde{G}_0^{\text{near}-1} & -e \\ -e & D_0^{\text{near}-1} \end{pmatrix}. \quad (\text{D28})$$

APPENDIX E: FUNCTIONAL RENORMALIZATION GROUP IN THE CTP FORMALISM

In the original version of the renormalization group method one follows the evolution of coupling constants one-by-one in the framework of the perturbation expansion [67] and exact results might be achieved by resumming the perturbation expansion. The functional realization of the renormalization group [68, 69] is based on one-loop evolution equation and the exact results should arise by letting infinitely many term mixing in the action. In this Appendix we outline briefly the evolution equation for the effective action in the latter formalism which can easily be converted into a numerical, non-perturbative algorithm to solve models.

Let us introduce a cutoff k in the theory for the field variable ϕ in the quadratic part of the action, $S[\phi] \rightarrow S[\phi] + i\phi \cdot K_k \cdot \phi/2$. The properties of the real operator K are: (i) It should suppress all fluctuations for $k \rightarrow \infty$, ie. the eigenvalues of K_∞ should be ∞ . (ii) The physical theory should be recovered for $k = 0$ which is guaranteed by the condition $K_0 = 0$. (iii) For finite, non-vanishing k K_k should suppress the modes with momentum below k ie. the eigenvalues of the translation invariant K_k should be large for momenta $p < k$. The generating functional for the connected Green functions of the field is

$$e^{iW[j^+, j^-]} = \int D[\hat{\phi}] e^{iS[\hat{\phi}] - \frac{1}{2}\hat{\phi} \cdot \hat{K} \cdot \hat{\phi} + i\hat{j} \cdot \hat{\phi}}, \quad (\text{E1})$$

$\hat{\phi} = (\phi^+, \phi^-)$ and $S[\hat{\phi}] = S[\phi^+] - S^*[\phi^-]$. The generating functional is equipped with UV regulator which is omitted for simplicity. The quadratic term in the exponent controls the fluctuations and

$$\hat{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}. \quad (\text{E2})$$

The evolution of the generating functional

$$\dot{W} = \frac{1}{2} \text{Tr} \left[\left(\frac{\delta^2 W}{\delta j^+ \delta j^+} + \frac{\delta^2 W}{\delta j^- \delta j^-} + i \frac{\delta W}{\delta j^+} \frac{\delta W}{\delta j^+} + i \frac{\delta W}{\delta j^-} \frac{\delta W}{\delta j^-} \right) \cdot \hat{K} \right] \quad (\text{E3})$$

is obtained by bringing the derivation with respect to k , denoted by a dot, into the functional integral. The parameterization (14) of the external sources gives

$$\dot{W} = \text{Tr} \left[\left(\frac{\delta^2 W}{\delta j \delta \bar{j}} - \kappa \frac{\delta^2 W}{\delta j \delta \bar{j}} + \frac{1 + \kappa^2}{4} \frac{\delta^2 W}{\delta \bar{j} \delta j} + \frac{i}{2} (\phi^+ \phi^+ + \phi^- \phi^-) \right) \cdot \dot{K} \right]. \quad (\text{E4})$$

We need the evolution of the effective action

$$\Gamma[\phi, \phi^{\text{im}}] = \Gamma[\phi, \phi^{\text{adv}}] + i \Im \Gamma[\phi^{\text{im}}, \phi^{\text{adv im}}] \quad (\text{E5})$$

where we separate the real and imaginary parts

$$\begin{aligned} \Gamma[\phi, \phi^{\text{adv}}] &= \Re W[j, \bar{j}] - \bar{j} \cdot \phi^{\text{adv}} - j \cdot \phi, \\ \Im \Gamma[\phi^{\text{im}}, \phi^{\text{adv im}}] &= \Im W[j, \bar{j}] - \bar{j} \cdot \phi^{\text{adv im}} - j \cdot \phi^{\text{im}} \end{aligned} \quad (\text{E6})$$

and introduce the variables

$$\begin{aligned} \phi + i\phi^{\text{im}} &= \frac{\delta W[j, \bar{j}]}{\delta j}, \\ \phi^{\text{adv}} + i\phi^{\text{adv im}} &= \frac{\delta W[j, \bar{j}]}{\delta \bar{j}}. \end{aligned} \quad (\text{E7})$$

The sources are real and therefore, two field variables are independent only. We shall use ϕ, ϕ^{adv} as independent variables and introduce the functional $\Gamma^{\text{im}}[\phi, \phi^{\text{adv}}] = \Im \Gamma[\phi^{\text{im}}, \phi^{\text{adv im}}]$ for the imaginary part. The derivation of the evolution equation for the effective actions is straightforward except that we have to keep track of the evolution of the relations among the independent and the dependent field variables. The expressions which follow will appear simpler by the help of the notations $\hat{\phi} = (\phi, \phi^{\text{adv}})$, $\hat{\phi}^{\text{im}} = (\phi^{\text{im}}, \phi^{\text{adv im}})$ adopted in the rest of this Appendix.

The equation (D1) allows us to rewrite the evolution equation (E4) in terms of the effective action,

$$\begin{aligned} \dot{\Gamma} = \text{Tr} \left\{ \left[\left(\frac{\delta^2 \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1}_{\phi\phi} - \kappa \left(\frac{\delta^2 \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1}_{\phi\phi^{\text{adv}}} + \frac{1 + \kappa^2}{4} \left(\frac{\delta^2 \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1}_{\phi^{\text{adv}}\phi^{\text{adv}}} \right. \right. \\ \left. \left. + i \left(\frac{\delta^2 \Im \Gamma}{\delta \hat{\phi}^{\text{im}} \delta \hat{\phi}^{\text{im}}} \right)^{-1}_{\phi^{\text{im}}\phi^{\text{im}}} - \kappa i \left(\frac{\delta^2 \Im \Gamma}{\delta \hat{\phi}^{\text{im}} \delta \hat{\phi}^{\text{im}}} \right)^{-1}_{\phi^{\text{im}}\phi^{\text{adv im}}} + i \frac{1 + \kappa^2}{4} \left(\frac{\delta^2 \Im \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1}_{\phi^{\text{adv im}}\phi^{\text{adv im}}} + \frac{i}{2} (\phi^+ \phi^+ + \phi^- \phi^-) \right] \cdot \dot{K} \right\}. \end{aligned} \quad (\text{E8})$$

where the subscript indicates block-matrix elements of the second functional derivative. Furthermore the disconnected contributions to the two-point functions are given in terms of $\phi^\pm = \phi + i\phi^{\text{im}} - (\kappa \mp 1)(\phi^{\text{adv}} + i\phi^{\text{adv im}})/2$.

The evolution equation (E8) is not yet closed because it contains the dependent field variables, too. To relate $\hat{\phi}^{\text{im}}$ and $\hat{\phi}$ locally in the field configuration space we introduce the derivative matrix

$$\hat{S}_{ab}[\hat{\phi}] = \frac{\delta \hat{\phi}_a}{\delta \hat{\phi}_b^{\text{im}}}. \quad (\text{E9})$$

The elimination of the variables $\hat{\phi}^{\text{im}}$ is achieved by the help of the derivation of the equations of motion

$$-\hat{j} = \frac{\delta \Im \Gamma}{\delta \hat{\phi}^{\text{im}}} = \frac{\delta \Gamma}{\delta \hat{\phi}} \quad (\text{E10})$$

with respect to $\hat{\phi}^{\text{im}}$,

$$\frac{\delta^2 \Im \Gamma}{\delta \hat{\phi}^{\text{im}} \delta \hat{\phi}^{\text{im}}} = \frac{\delta^2 \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \cdot \hat{S}. \quad (\text{E11})$$

The derivation of this equation with respect to λ yields the evolution equation for \hat{S} ,

$$\dot{\hat{S}}_{ab} = \left(\frac{\delta^2 \Gamma}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1}_{ac} \left[\left(\frac{\delta^2 \dot{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} \right)_{cd} \hat{S}_{db} - \hat{S}_{dc} \left(\frac{\delta^2 \dot{\Gamma}^{\text{im}}}{\delta \hat{\phi} \delta \hat{\phi}} \right)_{de} \hat{S}_{eb} - \frac{\delta \dot{\Gamma}^{\text{im}}}{\delta \hat{\phi}_d} \frac{\delta \hat{S}_{dc}}{\delta \hat{\phi}_e} (S^{-1})_{eb} \right]. \quad (\text{E12})$$

After the solution of the problem posed by the dependent variable we return to the evolution equation (E8) and we separate the tree-level contribution by the parameterization

$$\Gamma = \tilde{\Gamma} + i\Im\tilde{\Gamma} + \frac{i}{2}(\phi^+ \cdot \hat{K} \cdot \phi^+ + \phi^- \cdot \hat{K} \cdot \phi^-) \quad (\text{E13})$$

of the effective action. The real and imaginary parts of $\tilde{\Gamma}$ satisfy the evolution equations

$$\begin{aligned} \dot{\tilde{\Gamma}} &= \text{Tr} \left\{ \left[\left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} + A \right)_{\phi\phi}^{-1} - \kappa \left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} + A \right)_{\phi\phi\text{adv}}^{-1} + \frac{1+\kappa^2}{4} \left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} + A \right)_{\phi\text{adv}\phi\text{adv}}^{-1} \right] \cdot \dot{K} \right\}, \\ \dot{\tilde{\Gamma}}^{\text{im}} &= \text{Tr} \left\{ \left[\left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} \cdot \hat{S} + B \right)_{\phi\text{im}\phi\text{im}}^{-1} - \kappa \left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} \cdot \hat{S} + B \right)_{\phi\text{im}\phi\text{adv}\text{im}}^{-1} + \frac{1+\kappa^2}{4} \left(\frac{\delta^2 \Re \tilde{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} \cdot \hat{S} + B \right)_{\phi\text{adv}\text{im}\phi\text{adv}\text{im}}^{-1} \right] \cdot \dot{K} \right\}, \end{aligned} \quad (\text{E14})$$

respectively, where

$$\begin{aligned} A_{ab} &= \frac{1}{2} \left(\hat{K}_{ac} \frac{\delta \hat{\phi}_c^{\text{im}}}{\delta \hat{\phi}_b} + \frac{\delta \hat{\phi}_c^{\text{im}}}{\delta \hat{\phi}_a} \hat{K}_{cb} \right) + \hat{\phi}_c \hat{K}_{cd} \frac{\delta^2 \hat{\phi}_d^{\text{im}}}{\delta \hat{\phi}_a \delta \hat{\phi}_b}, \\ B_{ab} &= 2\hat{K}_{ab} - 2 \frac{\delta \hat{\phi}_c}{\delta \hat{\phi}_a^{\text{im}}} \hat{K}_{cd} \frac{\delta \hat{\phi}_d}{\delta \hat{\phi}_b^{\text{im}}} - \hat{\phi}_c \hat{K}_{cd} \frac{\delta^2 \hat{\phi}_d}{\delta \hat{\phi}_a^{\text{im}} \delta \hat{\phi}_b^{\text{im}}} - \frac{\delta^2 \hat{\phi}_c}{\delta \hat{\phi}_a^{\text{im}} \delta \hat{\phi}_b^{\text{im}}} \hat{K}_{cd} \hat{\phi}_d. \end{aligned} \quad (\text{E15})$$

The effective action of the physical system is obtained by integrating the system of equations (E12), (E14) from $k \approx \infty$ where a perturbative initial condition is imposed down to $k = 0$. The renormalized trajectory displays the scale dependence of the effective action. In the case of QED one should start with properly regulated theory eg. the effective action should be calculated as in Eq. (82) after the replacement $\hat{D}_0^{-1} \rightarrow \hat{D}_0^{-1} + T\hat{K}_\lambda$ in the action for the initial value of λ but keeping the original, λ -independent counterterms. The projection T into the transverse photon states in the suppression term renders the turning on of the interactions of photons with the fermion loops according to their momentum in a gauge invariant manner.

-
- [1] J. Schwinger, J. Math. Phys. **2**, 407 (1961); *Particles and Sources*, vol. I., II., and III., Addison-Wesley, Cambridge, Mass. 1970-73.
 - [2] R. D. Jordan, Phys. Rev. **D33**, 444 (1986).
 - [3] K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962); P. M. Bakshi, K. T. Mahanthappa, J. of Math. Phys. **4**, 1 (1963).
 - [4] R. A. Craig, J. of Math. Phys. **9**, 605 (1968).
 - [5] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)).
 - [6] L. Leplae, H. Umezawa, F. Manzini, Phys. Rep. **10**, 151 (1974).
 - [7] G. Zhou, Z. B. Su, B. Hao, L. Yu, Phys. Rep. **118**, 1 (1985).
 - [8] Z. B. Su, L. Y. Chen, X. T. Yu, K. C. Chou, Phys. Rev. **B37**, 9810 (1988).
 - [9] B. S. DeWitt, in *Quantum Concepts in Space and Time*, R. Penrose, C. J. Isham, eds. Clarendon, Oxford, 1986.
 - [10] T. Arimitsu, H. Umezawa, Prog. Theor. Phys. **77**, 32 (1987).
 - [11] H. Umezawa, *Advanced Field Theory: Micro, Macro and Thermal Physics* Am. Inst. of Phys. 1993.
 - [12] A. Campos, E. Verdaguier, Phys. Rev. **D49**, 1861 (1994).
 - [13] Y. Aharonov, P. G. Bergmann, J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).
 - [14] B. Reznik, Y. Aharonov, Phys. Rev. **A52**, 2538 (1995).
 - [15] R. P. Feynman, F. L. Vernon, Ann. Phys. **24**, 118 (1963).
 - [16] A. O. Caldeira, A. J. Leggett, Physica **121A**, 587 (1983); Ann. Phys. **149**, 374 (1983).
 - [17] W. Zwerger, Phys. Rev. **A31**, 1745 (1985).
 - [18] V. Hakim, V. Ambegoakar, Phys. Rev. **A32**, 423 (1985).
 - [19] H. Grabert, P. Schramm, G. L. Ingold, Phys. Rep. **168**, 115 (1988).
 - [20] E. Calzetta, A. Roura, E. Verdaguier, Physica **A317**, 188 (2003).
 - [21] H. D. Zeh, Found. Phys. **1**, 69 (1970).
 - [22] W. H. Zurek, Phys. Rev. **D26**, 1862 (1982).
 - [23] E. Joos, H. D. Zeh, Z. Phys. **B59**, 223 (1985).
 - [24] W. H. Zurek, Phys. Today **44**, 36 (1991).
 - [25] R. Griffiths, J. Stat. Phys. **36**, 219 (1984).
 - [26] R. Omnès, J. Stat. Phys. **53**, 893 (1988); Rev. Mod. Phys. **64**, 339 (1992).
 - [27] M. Gell-Mann, J. B. Hartle, Phys. Rev. **D47**, 3345 (1993).

- [28] B. L. Hu, J. P. Paz, Y. Zhang, Phys. Rev. **D45**, 2843 (1992); Phys. Rev. **D47**, 1576 (1993).
- [29] W. H. Zurek, J. P. Paz, S. Habib, Phys. Rev. Lett. **70**, 1187 (1993).
- [30] J. P. Paz, S. Habib, W. H. Zurek, Phys. Rev. **D47**, 488 (1993).
- [31] R. D. Jordan, Phys. Rev. **D36**, 3593 (1987); Phys. Rev. **D36**, 3604 (1987).
- [32] E. Calzetta, B. L. Hu, Phys. Rev. **D35**, 495 (1987).
- [33] J. P. Paz, Phys. Rev. **D40**, 1054 (1990); Phys. Rev. **D42**, 2 (1990).
- [34] E. Calzetta, B. L. Hu, Phys. Rev. **D40**, 656 (1989).
- [35] E. Calzetta, B. L. Hu, Phys. Rev. **D37**, 2878 (1988).
- [36] E. Calzetta, S. Habib, B. L. Hu, Phys. Rev. **D37**, 28 (1988).
- [37] E. Calzetta, B. L. Hu, Phys. Rev. **D61**, 025012 (1999).
- [38] J. Koide, Phys. Rev. **E62**, 5953 (2000).
- [39] D. A. R. Dalvit, F. D. Mazzitelli, Phys. Rev. **D54**, 6338 (1996).
- [40] J. Zanella, E. Calzetta, Phys. Rev. **E66**, 036134 (2002).
- [41] R. Haag, N. M. Hugenholtz, M. Winnink, Comm. Math. Phys. **5**, 215 (1967).
- [42] H. Araki, E. J. Woods, J. Math. Phys. **4**, 637 (1963).
- [43] H. Araki, W. Wyss, Helv. Phys. Acta **37**, 136 (1964).
- [44] N. P. Landsman, Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).
- [45] N. P. Landsman, Ch. G. van Weert, Ann. Phys. **186**, 141 (1988).
- [46] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1983).
- [47] C. Itzykson, J. B. Zuber, *Quantum Field Theory*, Chapter 7-1-1, McGraw-Hill, 1980.
- [48] K. Schwarzschild, Göttinger Nachrichten **128**, 132 (1903).
- [49] H. Tetrode, Zeits. f. Physik **10**, 317 (1922).
- [50] A. D. Fokker, Zeits. f. Physik **58**, 386 (1929); Physica **9**, 33 (1929); Physica **12**, 145 (1932).
- [51] J. A. Wheeler, R. Feynman, Rev. Mod. Phys. **21**, 425 (1949).
- [52] P. Hohenberg and W. Kohn, Phys. Rev. **136**, 864 (1964).
- [53] W. Kohn and L.J. Sham, Phys. Rev. **A140**, 1133 (1965).
- [54] E. H. Lieb, in *Density Functional Methods in Physics*, eds. R. M. Dreizler, J. da Providencia, Plenum Press, New York, 1985.
- [55] Nalewajski, J. Chem. Phys. **78**, 6112 (1983)
- [56] R. Fukuda, T. Kotani, S. Yokojima, Prog. Theor. Phys. **92**, 883 (1994).
- [57] R. Fukuda, M. Komachiya, S. Yokojima, Y. Suzuki, K. Okumura, T. Inagaki, Prog. Theor. Phys. Suppl. **121**, 1 (1995).
- [58] A. K. Rajagopal, F. A. Buot, in *Density Functional Theory, Topics in Current Chemistry*, ed. R. F. Nalewajski, Springer Verlag, 1996.
- [59] A. K. Rajagopal, Phys. Rev. **A54**, 3916 (1996).
- [60] M. Valiev, G. W. Fernando, *Generalized Kohn-Sham Density Functional Theory via Effective Action Formalism*, arXiv:cond-mat/9702247.
- [61] J. Polonyi, K. Sailer, Phys. Rev. **B66**, 155113 (2002).
- [62] G. Vignale, M. Rasolt, Phys. Rev. Lett. **59**, 2360 (1987); Phys. Rev. **B37**, 10685 (1988).
- [63] J. Polonyi, Phys. Rev. **B68**, 014202 (2003).
- [64] H. D. Zeh, *The Direction of Time*, Springer-Verlag, Berlin, 1989.
- [65] J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, 1975.
- [66] E. A. Uehling, Phys. Rev. **21**, 434 (1949).
- [67] K.G. Wilson, J. Kogut, Phys. Rep. **C12**, 77 (1974); K.G. Wilson, Rev. Mod. Phys. **47**, 773 (1975); Rev. Mod. Phys. **55**, 583 (1983).
- [68] C. Wetterich, Phys. Lett. **B301**, 90 (1993).
- [69] T. Morris, Int. J. Mod. Phys. **A9**, 2411 (1994).
- [70] J. Alexandre, V. Branchina, J. Polonyi, Phys. Rev. **D58**, 16002 (1988).
- [71] J. A. Wheeler, R. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
- [72] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, MacMillen Co., New-York, 1964.
- [73] F. Rohrlich, *Classical Charged Particles*, Addison-Wesley, Reading, Massachusetts 1965.
- [74] L. Landau, E. Lifshitz, *Classical Theory of Fields*, Addison-Wesley, Cambridge, Mass. 1951.